

Seiberg-Witten Theoretic Invariants of Lens Spaces

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Abstract

We describe an effective algorithm for computing Seiberg-Witten invariants of lens spaces. We apply it to two problems: (i) to compute the Froyshov invariants of a large family of lens spaces; (ii) to show that the knowledge of the Seiberg-Witten invariants of a lens space is topologically equivalent to the knowledge of its Casson-Walker invariant and of its Milnor-Turaev torsion.

Problem (i) has several interesting topological consequences concerning the negative definite manifolds bounding a given lens space.

Key words: *lens spaces, rational homology spheres, Seiberg-Witten equations and invariants, eta invariants, Froyshov invariants, Casson-Walker invariant, Milnor-Turaev torsion, Dedekind-Rademacher sums.*

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Introduction

The Seiberg-Witten theory of rational homology spheres is particularly difficult since the usual count of monopoles leads to a metric *dependent* integer. W. Chen [3], Y. Lim [15] and M. Marcolli [18] have shown that this count, suitably altered by a certain combination of eta invariants, leads to a topological invariant. For integral homology spheres, there is an unique $spin^c$ structure and this altered count was shown to coincide with the Casson invariant; see [4], [16], and [24] in the special case of Brieskorn spheres. For a rational homology sphere N there are $\#H_1(N, \mathbb{Z})$ such invariants which are rational numbers. They define a function

$$\mathbf{sw} : Spin^c(N) \rightarrow \mathbb{Q}, \quad \sigma \mapsto \mathbf{sw}(\sigma).$$

We will call \mathbf{sw} the Seiberg-Witten invariant of N . This invariant can be further formalized as follows.

Recall that $H_1(N, \mathbb{Z}) \cong H^2(N, \mathbb{Z})$ acts freely and transitively on the space $Spin^c(N)$ of $spin^c$ structures on N

$$Spin^c(N) \times H_1(N, \mathbb{Z}) \ni (\sigma, h) \mapsto \sigma \cdot h \in Spin^c(N)$$

Thus each $\sigma_0 \in Spin^c(N)$ defines an element $\mathbf{SW}_{\sigma_0} \in \mathbb{Q}[H]$ (= the rational group algebra of the multiplicative group $H = H_1(N, \mathbb{Z})$) defined by

$$\mathbf{SW}_{\sigma_0} = \sum_{h \in H} \mathbf{sw}(\sigma_0 \cdot h)h.$$

Clearly

$$\mathbf{SW}_{\sigma_0 \cdot g} = \mathbf{SW}_{\sigma_0} \cdot g^{-1}, \quad \forall g \in H.$$

Thus, the collection $\mathbf{SW} := \{\mathbf{SW}_{\sigma}; \sigma \in Spin^c(N)\} \subset \mathbb{Q}[H]$ coincides with an orbit of the right action of H on $\mathbb{Q}[H]$ so that the Seiberg-Witten invariant can be viewed as an element in $\mathbb{Q}[H]/H$.

This Seiberg-Witten invariant is unchanged by natural involution $\bar{\cdot} : Spin^c(N) \rightarrow Spin^c(N)$, $\sigma \mapsto \bar{\sigma}$. The conditions $\mathbf{sw}(\sigma) = \mathbf{sw}(\bar{\sigma})$ and $\sigma \cdot h = \bar{\sigma} \cdot h^{-1}$ imply

$$\mathbf{SW}_{\bar{\sigma}} = \overline{\mathbf{SW}_{\sigma}}$$

where $\bar{\cdot} : \mathbb{Q}[H] \rightarrow \mathbb{Q}[H]$ is the involution determined by $H \ni h \mapsto h^{-1} \in H$.

A few years ago, using Seiberg-Witten theory, Kim Froyshov ([8]) defined another invariant of a rational homology 3-sphere N which contains nontrivial information about the possible negative definite 4-manifolds which can bound N . His invariant is the sum of a highly unorthodox count of solutions of the Seiberg-Witten equations and the same combination of eta invariants entering the definition of \mathbf{sw} . This is done for each $spin^c$ structure and then, the maximum amongst these numbers is chosen. The resulting rational number is still metric dependent. To get rid of this dependence Froyshov takes the infimum over all “reasonable” metrics on N .

In [24] we have explicitly computed the invariant \mathbf{SW} for Brieskorn homology spheres with at most 4 singular fibers and we have identified it with the Casson invariant. In [22] we computed Froyshov’s invariant for many Brieskorn homology spheres with 3 singular fibers and we have indicated an algorithm for producing upper estimates for any Brieskorn sphere with three singular fibers.

In the present paper we use the results and techniques of [22] to produce a simple algorithm computing the \mathbf{SW} and the Froyshov invariants of lens spaces. As in [22], these formulæ involve the Dedekind-Rademacher sums so, each concrete computation, although completely elementary, can be quite involved. On the positive side, these computations can be easily implemented on any computer algebra system (such as *MAPLE*) and the numerical experiments reveal very beautiful patterns (see (2.30), (2.33)-(2.37)) and §3.3. The concrete computations lead to interesting topological consequences. Here are some samples of them.

The lens space $L(2k+1, 1)$ bounds no smooth, even, negative definite 4-manifold while the lens space $L(4k+1, 2)$ bounds no smooth, even, negative definite 4-manifold X such that $H_1(X, \mathbb{Z})$ has no 2-torsion.

Using recent results of Paolo Lisca, [17], we can deduce some information about the fillable contact structures on lens spaces. The following is an immediate consequence of Lisca’s work and our computations.

$L(2k+1, 1)$ cannot be the contact boundary of any even, symplectic 4-manifold.

Denote by $\mathbf{SW}_{p,q}$ is the Seiberg-Witten invariant of $L(p, q)$. It is an element of $\mathbb{Q}[\mathbb{Z}_p]/\mathbb{Z}_p$ and we will regard it as a polynomial in one variable t satisfying $t^p = 1$.

The ring $\mathbb{Q}[\mathbb{Z}_p]$ is equipped with an augmentation map

$$\mathbf{aug} : \mathbb{Q}[\mathbb{Z}_p] \rightarrow \mathbb{Q}, \quad \sum_{k=0}^{p-1} a_k t^k \mapsto \sum_{k=0}^{p-1} a_k.$$

We prove in §3.2, Theorem 3.2 that

$$\mathbf{aug}(\mathbf{SW}_{p,q}) = CW(L(p, q)). \quad (0.1)$$

where CW denotes the Casson-Walker invariant (see [33]) of a rational homology sphere normalized as in [14].

Following [19] we introduce the polynomial

$$\Sigma = \sum_{k=0}^{p-1} t^k.$$

It can be used to define a projection

$$\mathbf{Proj} : \mathbb{Q}[\mathbb{Z}_p] \rightarrow \Lambda_p := \ker \mathbf{aug}, \quad R \mapsto R - \frac{\mathbf{aug}(R)}{p} \Sigma.$$

Set

$$T_{p,q} = \mathbf{Proj}(\mathbf{SW}_{p,q}) = \mathbf{SW}_{p,q} - \frac{CW(L(p, q))}{p} \Sigma.$$

We can regard $T_{p,q}$ as an element of Λ/\mathbb{Z}_p . If A, B are two “polynomials” in Λ_p then $A \sim B$ will signify $A = t^n B$ for some $n \in \mathbb{Z}$.

The Milnor torsion of $L(p, q)$, which we denote by $\tau_{p,q}$, is also an element of Λ_p (see [19]). More precisely, using the convention of [31] we have (see [19], [31])

$$\tau_{p,q} \sim (1-t)^{-1}(1-t^q)^{-1}$$

i.e.

$$\tau_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}} := 1 - \frac{1}{p} \Sigma.$$

As explained in [19] the “polynomial” $\hat{\mathbf{1}}$ represents $\mathbf{1}$ in Λ_p . We prove the following.

For any positive integers p, q such that $\gcd(p, q) = \gcd(p, q-1) = 1$ we have

$$T_{p,q}(1-t)(1-t^q) \sim \tau_{p,q} \quad (0.2)$$

The restriction $\gcd(p, q-1) = 1$ is purely for technical reasons, to slightly simplify certain accounting jobs. The method we present works in the general case, when $\gcd(p, q-1) \geq 1$. We did not consider the details to be very enlightening so we have not included them. We will present them elsewhere. Instead, we present the results of some numerical experiments confirming the equality $T_{p,q} \sim \tau_{p,q}$ in the general case. The equality (0.2) confirms a hypothesis formulated in [32].

The paper consists of three parts and an Appendix. The first part is a review of basic, known facts about Seifert manifolds. Its inclusion in the present version of the paper is justified only by my constant worry to get all the signs right. The existent literature can be quite confusing and/or incomplete about the various orientation conventions.

The second part deals with the Froyshov invariants. The computational heart of the paper is §2.2 while the applications are collected in §2.3. §2.4 contains a number of conjectures concerning the Froyshov invariants suggested by numerical experiments. The most conceptual one loosely states that if the rational homology sphere N is the link of an isolated complex singularity then the “most complicated” negative definite which bounds N is the minimal resolution of the singularity.

The measure of complexity of a negative definite intersection form is given by the Elkies invariant described in §2.1. The third part is devoted to the proof of (0.1) and (0.2).

Acknowledgements I want to thank Paolo Lisca for his interest in these issues. It was his paper [17] and his e-mail questions on the Froyshov invariants which attracted my attention to lens spaces. I learned about the Seiberg-Witten invariants of rational homology spheres from Weimin Chen who is one of the pioneers in this subject and I want to thank him for the useful conversations over the years. As always, Nikolai Saveliev has generously shared his knowledge with me. In particular, he made me aware of [2] which provided the stimulus for the third part of this paper. I am indebted to Frank Connolly for patiently explaining the Whitehead torsion to me and in general, for the many helpful mathematical conversations. Finally, I want to thank Yuhun Lim for sending me his preprints [15] and [16].

Orientation conventions Throughout this paper we will use the following orientation conventions.

- The boundary of an oriented manifold M is given the outer-normal-first orientation i.e.

$$\mathbf{or}(M) = \text{outer normal} \wedge \mathbf{or}(\partial M).$$

- The total space of a fibration $F \hookrightarrow E \rightarrow B$ is given the fiber-first orientation i.e.

$$\mathbf{or}(E) = \mathbf{or}(F) \wedge \mathbf{or}(B).$$

If E is (locally) a principal S^1 bundle then the fibers are given the orientation induced by the action of S^1 .

- If σ is a $spin^c$ structure on an oriented Riemann 3-manifold (N, g) and \mathbb{S}_σ is the associated bundle of complex spinors then the Clifford multiplication $\mathbf{c} : \Omega^*(N) \rightarrow \text{End}(\mathbb{S}_\sigma)$ is chosen such that

$$\mathbf{c}(*1) = -\text{Id}.$$

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1 A review of Seifert fibrations

The goal of this section is to survey existing results concerning Seifert fibration and, in particular, clarify the many orientation conventions concerning the Seifert invariants.

§1.1 Classification results In this paper, a *Seifert manifold (or fibration)* is a compact, *oriented*, smooth 3-manifold N without boundary, equipped with an infinitesimally free S^1 action. The orbits of the S^1 -action are called *fibers*. A fiber $S^1 \cdot x$ is called *regular* if the stabilizer \mathbf{St}_x of x is trivial. Otherwise, the fiber is called *singular*. In this case \mathbf{St}_x is a cyclic group \mathbb{Z}_α and the order of this stabilizer is called the *multiplicity* of the fiber. It is customary to identify \mathbf{St}_x with the cyclic subgroup

$$C_\alpha = \{\exp(\frac{2k\pi i}{\alpha}); k = 0, 1, \dots, \alpha - 1\} \subset S^1.$$

For brevity set $\rho_\alpha := \exp(\frac{2\pi i}{\alpha})$. The *base* of the Seifert fibration is the space of orbits $\Sigma := N/S^1$. Topologically, it is a compact oriented surface but smoothly, it is a 2-dimensional orbifolds. The orbifold singularities are all cone-like and correspond bijectively to the singular fibers.

Equip N with an S^1 -invariant Riemann metric h . Suppose $F \subset N$ is a singular fiber of multiplicity α containing the point x . The bundle $TN|_F$ splits orthogonally as

$$TN|_F = TF \oplus (TF)^\perp.$$

Both TF and $(TF)^\perp$ are S^1 -equivariant bundles over F . The stabilizer C_α of x acts *effectively* on $(T_x F)^\perp$. Denote this action by

$$\tau : C_\alpha \rightarrow \text{Aut}((T_x F)^\perp).$$

If we identify $(T_x F)^\perp$ as an oriented vector space with \mathbb{C} then τ is completely described by an integer $0 < q < \alpha$, $\text{g.c.d.}(q, \alpha) = 1$ by the formula

$$\tau(\rho_\alpha)z = \rho_\alpha^q z.$$

We will denote this action by $\tau_{\alpha,q}$ or, when no confusion is possible, by τ_q . Following [25] we call the pair (α, q) the *orbit invariant* of the singular fiber F . Now denote with β the integer uniquely determined by the requirements

$$0 < \beta < 1, \quad \beta q \equiv 1 \pmod{\alpha}.$$

The pair (α, β) is called the (*oriented, normalized*,) *Seifert invariant* of the singular fiber F .

Using the principal C_α -bundle

$$P_\alpha = (S^1 \rightarrow S^1), \quad z \mapsto z^\alpha$$

and the representation τ_q we can form the associated S^1 -equivariant line bundle

$$E_{\alpha,q} := P_\alpha \times_{\tau_q} \mathbb{C} \rightarrow S^1.$$

The S^1 -action on $E_{\alpha,q}$ is induced from the obvious action on $S^1 \times \mathbb{C}$

$$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, z_2), \quad |z_1| = 1, \quad z_2 \in \mathbb{C}$$

which commutes with the action of C_α

$$\rho_\alpha(z_1, z) = (\rho_\alpha z_1, \rho_\alpha^{-q} z_2).$$

To describe this more explicitly note first that $E_{\alpha,q}$ is diffeomorphic to $S^1 \times \mathbb{C}$. Such an diffeomorphism can be obtained using the C_α invariant map

$$T : S^1 \times \mathbb{C} \rightarrow S^1 \times \mathbb{C}, (z_1, z_2) \xrightarrow{T} (\zeta_1, \zeta_2) = (z_1^\alpha, z_1^q z_2).$$

Then we can regard (ζ_1, ζ_2) as global coordinates on $E_{\alpha,q}$ and we can describe the S^1 -action by

$$e^{i\theta}(\zeta_1, \zeta_2) = T e^{i\theta} \cdot (z_1, z_2) = (e^{i\alpha\theta} \zeta_1, e^{iq\theta} \zeta_2).$$

We have the following result (see [25]).

The Slice Theorem *There exists an S^1 -invariant open neighborhood U of F in N , an S^1 -invariant open neighborhood V of the zero section of $E_{\alpha,q}$ and an S^1 -equivariant diffeomorphism*

$$\phi : V \rightarrow U$$

which maps the zero section to F and $\mathbf{1} \in S^1$ to a given fixed point $x \in F$.

Denote D_r denotes the disk of radius r in the fiber of $E_{\alpha,q}$ over $\mathbf{1} \in S^1$ i.e.

$$D_r = \{(1, \zeta_2) \in E_{\alpha,q}; |\zeta_2| \leq r\}.$$

The surface $\phi(D_r)$ will be called a *slice* of the S^1 -action. For simplicity, we will continue to denote it by D_r . Its boundary, equipped with the induced orientation, will be denoted by $\vec{\sigma}$. It can be explicitly described by the parameterization

$$(\zeta_1, \zeta_2) = (1, r e^{it}), \quad t \in [0, 2\pi].$$

Denote by $\Delta(r) = \Delta_{\alpha,\beta}$ the bundle of disks of radius r determined by $E_{\alpha,q}$ and set $S(r) = S_{\alpha,\beta} := \partial\Delta_{\alpha,\beta}$. $\Delta(r)$ is topologically a solid torus. It is usually known as the *fibered torus* corresponding to the Seifert invariants (α, β) . Endow $S(r)$ with the induced orientation. $S(r)$ is equipped with a *free* S^1 -action. Denote by \vec{f} such an orbit, endowed with the induced orientation. It can be described explicitly by the curve

$$(\zeta_1, \zeta_2) = (e^{i\alpha t}, e^{iq t}), \quad t \in [0, 2\pi].$$

\vec{f} meets $\vec{\sigma}$ geometrically α -times. In fact, with all the above orientation conventions in place, we also have $\vec{\sigma} \cdot \vec{f} = \alpha$, algebraically as well.

A *section* of the S^1 -action on $S(r)$ is a closed, oriented curve \vec{s} such that $\vec{s} \cdot \vec{f} = 1$ both algebraically and geometrically. There exist many sections. We want to show that there exists a section satisfying the homological condition

$$\vec{\sigma} = \alpha \vec{s} + \beta \vec{f}. \tag{1.1}$$

Clearly the above condition uniquely determines the homology class of \vec{s} in S_r .

To find a section satisfying (1.1) we first choose a longitude, i.e. a simple, closed, oriented curve $\vec{\lambda}$ such that $\vec{\sigma} \cdot \vec{\lambda} = 1$. There is no unique choice, but two choices differ by a multiple of $\vec{\sigma}$. Note

that the image of such a $\vec{\lambda}$ in $H_1(\Delta(r), \mathbb{Z})$ coincides with the positive generator, or via ϕ , with the singular fiber F . Then

$$\vec{f} = u\vec{\sigma} + v\vec{\lambda}, \quad u, v \in \mathbb{Z}$$

and since $\vec{\sigma} \cdot \vec{f} = \alpha$ we deduce $v = \alpha$ i.e.

$$\vec{f} = u\vec{\sigma} + \alpha\vec{\lambda}.$$

Since \vec{f} “wraps” along $\vec{\sigma}$ q -times, the coordinate u is uniquely determined mod α , more precisely $u \equiv q \bmod \alpha$. Now choose $\vec{\lambda}$ so that $u = q$ i.e.

$$\vec{f} = q\vec{\sigma} + \alpha\vec{\lambda}_0. \tag{1.2}$$

We call $\vec{\lambda}$ the *canonical longitude*. The sought for section \vec{s} has a decomposition

$$\vec{s} = x\vec{\sigma} + y\vec{\lambda}$$

subject to the constraint (1.1) which becomes

$$\vec{\sigma} = (x\alpha + \beta q)\vec{\sigma} + (\beta\alpha + \alpha y)\vec{\lambda}.$$

Since $\beta q \equiv 1 \bmod \alpha$, there exists an unique pair (x_0, y_0) so that the above equality is satisfied. More precisely

$$x_0 = (1 - \beta q)/\alpha, \quad y_0 = -\beta.$$

Thus the *canonical section*, determined by (1.1) is

$$\vec{s} = x_0\vec{\sigma} - \beta\vec{\lambda}. \tag{1.3}$$

We can now use these notions to describe the structure of Seifert fibrations. Suppose the Seifert fibration has $m \geq 1$ singular fibers F_{x_1}, \dots, F_{x_m} with normalized Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$. Delete small, pairwise disjoint, S^1 -invariant neighborhoods U_1, \dots, U_m of the singular fibers, determined by the Slice Theorem. We get a 3-manifold with boundary

$$N' = N \setminus \left(\bigcup_{i=1}^m U_i \right)$$

equipped with a free S^1 -action. This is a principal S^1 -bundle $S^1 \hookrightarrow N' \rightarrow \Sigma' := N'/S^1$. The restriction of this bundle to $\partial\Sigma'$ has canonical sections, determined by (1.1). In other words, it is trivialized along the boundary. Such a bundle is completely determined topologically by an integer b , the relative degree (or Euler number). Here we have to warn the reader that our b differ by a sign from the conventions in [13] or [20].

We can now reconstruct N from N' and the equivariant bundles E_{α_i, q_i} by attaching the fibered torus $\Delta_{\alpha_i, \beta_i}$ to the i -th boundary component $\partial_i N'$ of N' using the attaching rules (1.1)

$$\vec{\sigma}_i = \alpha_i \vec{s}_i + \beta_i \vec{f}_i, \quad i = 1, \dots, m.$$

We have to be very careful about the orientation conventions. More precisely, $(\vec{\sigma}_i, \vec{\lambda}_i)$ and (\vec{s}_i, \vec{f}_i) are bases of $H_1(\partial\Delta_{\alpha_i, \beta_i}(r), \mathbb{Z})$ compatible with the orientation of $\partial\Delta_{\alpha_i, \beta_i}$ regarded as boundary of $\Delta_{\alpha_i, \beta_i}$. Denote by $\vec{\mu}_i$ the i -th boundary component of Σ' oriented accordingly. We regard it as an oriented curve on $\partial N'$ via the above trivialization of $N' |_{\partial\Sigma'}$. Then $(\vec{\mu}_i, \vec{f}_i)$ is compatible

with the orientation of $\partial\Delta_{\alpha_i, \beta_i}$ regarded as a component of $\partial N'$. On the other hand, $\vec{\mu}_i = -\vec{s}_i$ in $H_1(\partial\Delta_{\alpha_i, \beta_i}, \mathbb{Z})$. The attaching map

$$\gamma_i : \partial_i N' \rightarrow \partial\Delta_{\alpha_i, \beta_i}$$

is given by the identifications (1.3) and (1.2)

$$\vec{\mu}_i = -\vec{s}_i \mapsto -x_i \vec{\sigma}_i + \beta_i \vec{\lambda}_i, \quad \vec{f}_i \mapsto q_i \vec{\sigma}_i + \alpha_i \vec{\lambda}_i \quad (1.4)$$

where $\alpha_i x_i + \beta_i q_i = 1$. If we choose angular coordinates (θ^1, θ^2) on $\partial_i N'$ and (φ^1, φ^2) on $\partial\Delta_{\alpha_i, \beta_i}$ such that $\vec{\mu}_i := (\theta^1 = t, \theta^2 = 0)$, and $\vec{s}_i = (\varphi^1 = t, \varphi^2 = 0)$, $t \in [0, 2\pi]$, then the above gluing map can be given the matrix description

$$\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \begin{bmatrix} -x_i & q_i \\ \beta_i & \alpha_i \end{bmatrix} \cdot \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix}$$

The above matrix has determinant -1 and inverse

$$\Gamma_{\alpha_i, \beta_i} := \begin{bmatrix} -\alpha_i & q_i \\ \beta_i & x_i \end{bmatrix}. \quad (1.5)$$

It is customary to regard the above procedure the opposite way, as attaching $\Delta_{\alpha_i, \beta_i}$ to $\partial_i N'$ via the orientation reversing map

$$\Gamma_{\alpha_i, \beta_i} : \partial\Delta_{\alpha_i, \beta_i} \rightarrow \partial_i N'.$$

Now denote by ℓ the rational number

$$\ell = b - \sum \frac{\beta_i}{\alpha_i}.$$

It is called the rational number of the Seifert fibration. The *normalized Seifert invariant* of N is defined as the collection

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$$

where g denotes the genus of Σ .

The above discussion shows that any Seifert manifold is uniquely determined (up to an S^1 -equivariant diffeomorphism) by its Seifert invariant. Moreover, given a collection as above (with obvious restrictions on the pairs (α_i, β_i)) one can construct a Seifert manifold with precisely this normalized Seifert invariant. To see this, we need only to explain how to construct an S^1 -bundle over a Riemann surface Σ' of genus g , obtained from a closed surface Σ by deleting m pairwise disjoint disks D_1, \dots, D_m . This construction proceeds as follows.

First, delete one more disk D_0 from Σ' which does not meet $\partial\Sigma'$. Set $\Sigma'' = \Sigma' \setminus D_0$ and $N'' = \Sigma'' \times S^1$. Denote by $\partial_0 \Sigma''$ the new boundary component. Now attach $D_0 \times S^1$ to $\partial_0 N''$ using the orientation reversing map

$$\Gamma_b : \partial D_0 \times S^1 \rightarrow \partial_0 N''$$

given by the matrix

$$\Gamma_b = \begin{bmatrix} -1 & 0 \\ -b & 1 \end{bmatrix}. \quad (1.6)$$

The S^1 -bundle obtained in this manner is trivialized along the boundary of Σ' and has relative degree b . (For $m = 0$ this construction mimics the construction of the holomorphic line bundle on Σ associated to the divisor bP , where P is the center of D_0 .) If we set $\vec{\sigma}_0 = \partial D_0$, $\vec{\lambda}_0 = \{1\} \times S^1 \subset$

$\partial D_0 \times S^1$, $\vec{\mu}_0 = \partial_0 \Sigma'' \times \{1\} \subset \partial_0 N''$ and $\vec{f}_0 = \{1\} \times S^1 \subset \partial_0 \Sigma'' \times S^1$, then the above gluing map produces the identifications

$$\mu_0 \mapsto -\vec{\sigma}_0 - b\vec{\lambda}_0, \quad \vec{f}_0 \mapsto \vec{\lambda}_0. \quad (1.7)$$

Often it is useful to work with un-normalized Seifert invariants. These are collections

$$\mathbf{S} = (g, b, m; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$$

such that $g.c.d(\beta_i, \alpha_i) = 1$, $\alpha_i \neq 0$. Two collections \mathbf{S} and \mathbf{S}' are called equivalent if $g = g'$, the collection of α_i -s not equal to 1 coincides (including multiplicities) with the collection of α'_j -s not equal to 1 and

$$b - \sum_i \frac{\beta_i}{\alpha_i} = b' - \sum_j \frac{\beta'_j}{\alpha'_j}.$$

Clearly, by choosing a section other than the canonical one, we arrive at an un-normalized Seifert invariant. We refer the reader to [13] or [25] for a proof of the fact that equivalent un-normalized Seifert invariants lead to S^1 -diffeomorphic Seifert manifolds.

Using the normalized Seifert invariants, and the above gluing description of a Seifert manifold, it is very easy to determine its fundamental group via Van Kampen's theorem. The fundamental group of N'' has generators

$$a_j, b_j, \vec{\mu}_i, \vec{f}, \quad 1 \leq j, \quad 0 \leq i \leq m$$

and relations

$$[a_1, b_1] \cdots [a_g, b_g] \vec{\mu}_0 \cdots \vec{\mu}_m = [a_j, \vec{f}] = [b_j, \vec{f}] = [\vec{\mu}_i, \vec{f}] = 1.$$

Attaching the solid torus $D_0 \times S^1$ we introduce a new relation given by (1.7) namely

$$\vec{\mu}_0 = \vec{f}^{-b}.$$

Attaching the fibered torus $\Delta_{\alpha_i, \beta_i}$ we introduce an additional generator, $\vec{\lambda}_i$ and additional relations, given by (1.4), namely

$$\vec{\mu}_i = \vec{\lambda}_i^{\beta_i}, \quad \vec{f} = \vec{\lambda}_i^{\alpha_i}.$$

Recall that $\vec{\mu}_i$ is a section of N'' over $\partial_i \Sigma''$ oriented as a boundary component of Σ'' and \vec{f} denotes the class of a regular fiber. $\vec{\lambda}_i$ can be expressed in terms of $\vec{\mu}_i$ and \vec{f} by $\vec{\lambda}_i = \vec{\mu}_i^{q_i} \vec{f}^{x_i}$ where $\alpha x_i + \beta_i q_i = 1$. Thus, attaching the fibered torus $\Delta_{\alpha_i, \beta_i}$ has the overall effect of introducing the relation

$$\vec{\mu}_i^{\alpha_i} = \vec{f}^{\beta_i}.$$

Thus the fundamental group of N can be given the presentation

- **Generators** a_j, b_j ($1 \leq j \leq g$), $\vec{\mu}_i$ ($1 \leq i \leq m$), \vec{f} .
- **Relations**

$$\vec{f}^{-b} [a_1, b_1] \cdots [a_g, b_g] \vec{\mu}_1 \cdots \vec{\mu}_m = [a_j, \vec{f}] = [b_j, \vec{f}] = [\vec{\mu}_i, \vec{f}] = \vec{\mu}_i^{\alpha_i} \vec{f}^{-\beta_i} = 1.$$

In [9] the Seifert manifolds were given a different interpretation in terms of V -line bundles over V -surfaces. This led to different Seifert invariants. We conclude this subsection with a description of the relationship between the Seifert invariants of [9] (or [22]) and the Seifert invariants used in this paper.

As explained in [9], there is an alternative procedure of obtaining all the Seifert manifolds. Start with a V -surface Σ with m singular points x_1, \dots, x_m with isotropies $C_{\alpha_1}, \dots, C_{\alpha_m}$. Pick a

complex line V -bundle $L \rightarrow \Sigma$ such that the isotropies in the fibers over the singular points are given by the representations

$$\tau_{\alpha_i, \omega_i} : C_{\alpha_i} \rightarrow U(1), \quad \tau_{\alpha_i, \omega_i}(\rho_{\alpha_i}) = \rho_{\alpha_i}^{\omega_i}.$$

Above, ω_i are integers satisfying the conditions

$$0 < \omega_i < \alpha_i, \quad g.c.d(\alpha_i, \omega_i) = 1. \quad (1.8)$$

Then the unit circle bundle $N = S(L)$ determined by L is a Seifert manifold. In [22] we defined the Seifert invariants as the collection

$$(g, \ell, m; (\alpha_1, \omega_1), \dots, (\alpha_m, \omega_m))$$

where ℓ is the rational degree of L . We will refer to these as the *Seifert V -invariants*.

We want to show that the normalized Seifert invariants (as defined in this paper) of N are

$$\beta_i := \alpha_i - \omega_i \quad (1.9)$$

and

$$b = \ell + \sum_i \frac{\beta_i}{\alpha_i}. \quad (1.10)$$

To establish these facts we have to understand the orbit invariants of the singular fibers of $S(L)$.

A neighborhood of the singular fiber of $S(L)$ sitting over the singular point $x = x_i$ can be described as the C_α -quotient of the C_α -equivariant S^1 -bundle

$$T_{\alpha, \omega} \rightarrow D = \{z \mid |z| < 1\} \subset \mathbb{C}$$

where C_α acts on $T_{\alpha, \omega}$ by

$$\rho_\alpha \cdot (z_1, z_2) = (\rho_\alpha z_1, \rho_\alpha^\omega z_2), \quad (|z_1| < 1, |z_2| = 1)$$

while S^1 acts by

$$e^{i\theta}(z_1, z_2) = (z_1, e^{i\theta} z_2).$$

Note that there exists a natural diffeomorphism

$$T_{\alpha, \omega}/C_\alpha \rightarrow D \times S^1$$

induced by the C_α -invariant map

$$A : D \times S^1 \rightarrow D \times S^1, \quad (z_1, z_2) \xrightarrow{A} (\zeta_1, \zeta_2) = (z_1 z_2^{-s}, z_2^\alpha)$$

where $s\omega \equiv 1 \pmod{\alpha}$. We see that $T_{\alpha, \omega}/C_\alpha$ admits an S^1 -action given by

$$e^{i\theta}(\zeta_1, \zeta_2) = A e^{i\theta}(z_1, z_2) = (e^{-is\theta} \zeta_1, e^{i\alpha\theta} \zeta_2).$$

Hence, the orbit invariants (α, q) of this action are $(\alpha, q) = (\alpha, -s)$. Thus $\omega q \equiv -1 \pmod{\alpha}$ so that $\omega \equiv -\beta \pmod{\alpha}$. The equality (1.9) now follows immediately from the normalization condition (1.8). We leave the equality (1.10) to the reader.

We want to clarify one point. Denote by $|L|$ the desingularization of L (described in [22]). Then

$$\deg |L| = \deg L - \sum_i \frac{\omega_i}{\alpha_i} = \ell + \sum_i \frac{\beta_i}{\alpha_i} - m = b - m. \quad (1.11)$$

The description of Seifert fibrations via line V -bundles has its computational advantages. It allows a very convenient description of the cohomology group $H^2(N, \mathbb{Z})$. We include it here for later use.

Consider a Seifert fibration N over a 2-orbifold Σ defined as the unit circle bundle determined by a line V -bundle $L_0 \rightarrow \Sigma$. Suppose the singularities of Σ have isotropies $\alpha_1, \dots, \alpha_m$ while the isotropies of L_0 over the singular points are described by $\rho_{\alpha_i}^{\omega_i}$ as explained above. Denote by $\text{Pic}^t(\Sigma)$ the space (Abelian group more precisely) of isomorphism classes of line V -bundles over Σ . Define a group morphism

$$\tau : \text{Pic}^t(\Sigma) \rightarrow \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \dots \oplus \mathbb{Z}_{\alpha_m}$$

by

$$\tau(L) = (\deg L, \gamma_1 \bmod \alpha_1, \dots, \gamma_m \bmod \alpha_m)$$

where $\deg L$ is the rational degree of L and γ_i describe the isotropies of L over the singular points of Σ . Next, define

$$\delta : \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \dots \oplus \mathbb{Z}_{\alpha_m} \rightarrow \mathbb{Q}/\mathbb{Z}$$

by

$$\delta(c, \gamma_1, \dots, \gamma_m) = \left(c - \sum_i \frac{\gamma_i}{\alpha_i} \right) \bmod \mathbb{Z}.$$

In [9] it is shown that the sequence below is exact

$$0 \rightarrow \text{Pic}^t(\Sigma) \xrightarrow{\tau} \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \dots \oplus \mathbb{Z}_{\alpha_m} \xrightarrow{\delta} \mathbb{Q}/\mathbb{Z} \rightarrow 0. \quad (1.12)$$

Moreover, there exists an isomorphism of groups

$$H^2(S(L_0), \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \text{Pic}^t(\Sigma)/\mathbb{Z}[L_0] \quad (1.13)$$

where g is the genus of Σ and $\mathbb{Z}[L_0]$ denotes the cyclic subgroup of $\text{Pic}^t(\Sigma)$ generated by L_0 . The subgroup $\text{Pic}^t(\Sigma)/\mathbb{Z}[L_0]$ of $H^2(S(L_0), \mathbb{Z})$ consists of the Chern classes of the line bundles on $S(L_0)$ obtained by pullback from line V -bundles on Σ .

§1.2 Plumbing description of Seifert fibrations The Seifert manifolds can be represented as boundaries of certain 4-manifolds naturally determined from the Seifert invariant. This is achieved via the *plumbing construction* which we proceed to describe in this section. Our presentation is greatly inspired from [11]. We have to warn the reader that our Seifert invariant conventions differ from those in [11]. Ours coincide (except for the sign of b) with those in [13] or [20].

We begin by introducing the plumbing construction and a simple way of visualizing it.

Consider two disk bundles $E_1 \rightarrow \Sigma_1$ and $E_2 \rightarrow \Sigma_2$ over the Riemann surfaces Σ_1 and Σ_2 , of degrees $\deg E_1 = b$ and $\deg E_2 = c$. These are 4-manifolds with boundaries circle bundles over Riemann surfaces. As bundles, they are determined by principal S^1 bundles over Σ_1 and Σ_2 of degrees b and respectively c . We present below a computationally friendly way of representing these principal S^1 -bundles (and thus the associated disk bundles as well). We proceed as follows.

- Pick a small disk $\Delta_1 \subset \Sigma_1$ and form the trivial S^1 -bundle $\Delta_1 \times S^1 \rightarrow \Delta_1$. Set $\Sigma'_1 = \Sigma_1 \setminus \Delta_1$
- Fix the integer r and then denote by $N'_1(r)$ the S^1 -bundle over Σ'_1 equipped with a trivialization over $\partial\Sigma'_1$ and having relative degree r .
- We can obtain a degree b S^1 -bundle over Σ_1 by attaching the solid torus $\Delta_1 \times S^1$ to $\partial N'_1(r)$ via the gluing map $\Gamma_{b-r} : \partial\Delta_1 \times S^1 \rightarrow \partial N'_1(r)$ described in (1.6).

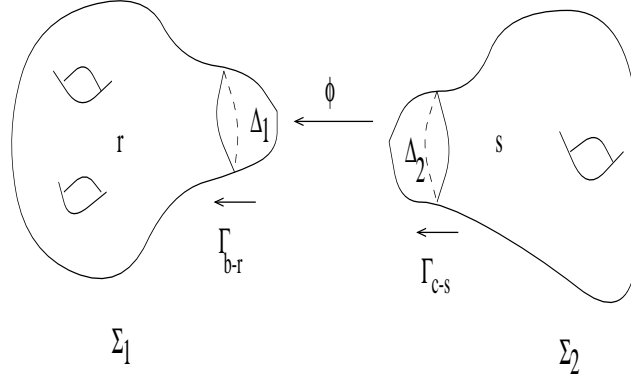


Figure 1: *Plumbing two disk bundles*

These three steps can be represented graphically as in the left-hand side of Figure 1. We can produce a similar description for E_2 in which b is replaced by c and, instead of r , we pick a different integer s . This is illustrated in the right-hand-side of Figure 1. We will refer to such a figure as the *diagram of a plumbing*. We will not indicate the integer r on the diagram when we chose it to be zero. The same convention applies for s .

To plumb the disk bundles E_1 and E_2 proceed as follows.

- Identify Δ_1 and Δ_2 with the unit disk D in the plane and fix trivializations $E_i|_{\Delta_i} \rightarrow D \times D$.
- Now glue $E_2|_{\Delta_2}$ onto $E_1|_{\Delta_1}$ using the gluing map

$$\phi : D \times D \rightarrow D \times D, \quad (z_1, z_2) \mapsto (z_2, z_1).$$

The resulting space $E_1 \#_\phi E_2$ has apparent corners which can be “smoothed-out” to produce a 4-manifold with boundary called the *plumbing of the two disk bundles*. Its boundary can be alternatively described as follows.

- Attach $\partial N'_2(s)$ to $\partial N'_1(r)$ using the sequence of gluings

$$\partial N'_2(s) \xrightarrow{\Gamma_{c-s}^{-1}} \partial \Delta_2 \times S^1 \xrightarrow{\phi} \partial \Delta_1 \times S^1 \xrightarrow{\Gamma_{b-r}} \partial N'_1(r).$$

Observe now that $\Gamma_d = \Gamma_d^{-1}$ for any $d \in \mathbb{Z}$. Thus, the boundary of the plumbing $E_1 \#_\phi E_2$ can be obtained by attaching $\partial N'_2(s)$ to $\partial N'_1(r)$ via the gluing map

$$\Gamma_{b-r} \circ \phi \circ \Gamma_{c-s}.$$

A *star-shaped graph* is a connected tree with a distinguished vertex v_0 (called the *center*) such that the degree of any vertex other than the center is ≤ 2 . A *branch* of a star-shaped graph is a connected component of a the graph obtained by removing the center. A *weight* on a star-shaped graph Γ is a map

$$w : \text{Vertex}(\Gamma) \rightarrow (\mathbb{Z}_+ \times \mathbb{Z}) \cup \mathbb{Z}$$

such that

$$w(\text{center}) \in \mathbb{Z}_+ \times \mathbb{Z}$$

and for any vertex $p \neq \text{center}$ $w(p) \in \mathbb{Z}$ (see Figure 2). A weighted star-shaped graph (Γ, w) encodes the following topological operations.

- If the weight of the center is (g, d) associate to it a disk bundle of degree d over a Riemann surface of genus g .

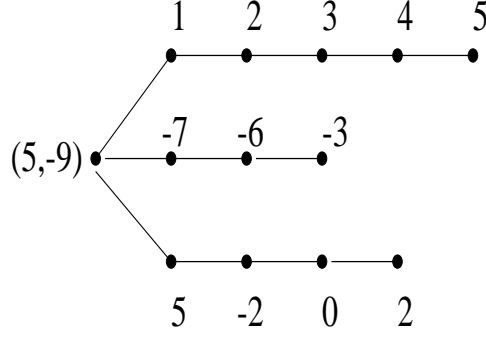


Figure 2: A weighted star-shaped graph

- To any vertex, other than the center, of weight n , associate a degree n disk bundle over S^2 .
- Plumb the above disk bundles following the edges of Γ i.e. two bundles are plumbed iff the corresponding edges are joined by an edge.

In this manner we obtain a 4-manifold with boundary $P(\Gamma, w)$. We have the following theorem of von Randow, [29]; see also [25].

Theorem 1.1 The boundary of $P(\Gamma, w)$ has a natural structure of Seifert manifold. The Seifert invariants can be read off the weighted graph (Γ, w) .

Let us describe how to read off an *un-normalized* Seifert invariant

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$$

of $\partial P(\Gamma, w)$. First of all the number m is precisely the number of branches of Γ . (g, b) is the weight of the center. Finally, if the weights on the i -th branch are w_{i1}, \dots, w_{ik} then the *irreducible* fraction α_i/β_i is recovered from the continuous fraction decomposition

$$\frac{\alpha_i}{\beta_i} = [w_{i1}, w_{i2}, \dots, w_{ik}]$$

where for any integers n_1, \dots, n_k with $n_k \neq 0$ we define inductively

$$[n_1, n_2, \dots, n_k] = n_1 - \frac{1}{[n_2, \dots, n_k]} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \frac{1}{\ddots - \frac{1}{n_k}}}}.$$

We check this on the simple graph depicted in Figure 3. The boundary of the plumbing is obtained by gluing the solid disk $D \times S^1$ (D is described in Figure 4) to the boundary of an S^1 -bundle of relative degree w_0 over a disk. The attaching map can be read easily from the diagram in Figure 4 and it is

$$\gamma = \Gamma_0 \circ (\phi \circ \Gamma_{w_1}) \circ (\phi \circ \Gamma_{w_2}) \circ (\phi \circ \Gamma_{w_3}).$$

Set

$$S_b := \phi \circ \Gamma_b = \begin{bmatrix} -b & 1 \\ -1 & 0 \end{bmatrix}.$$

We deduce

$$\phi \circ \gamma = S_0 \circ S_{w_1} \circ S_{w_2} \circ S_{w_3} = \begin{bmatrix} -w_2 w_3 + 1 & * \\ w_1 w_2 w_3 - w_1 - w_3 & * \end{bmatrix}.$$

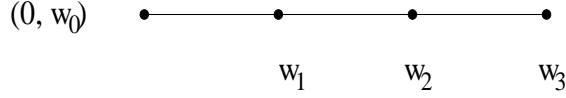


Figure 3: A simple plumbing

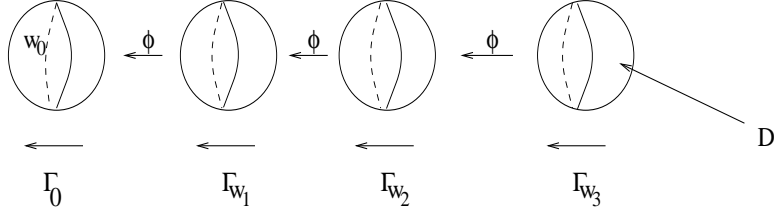


Figure 4: The diagram of a simple plumbing

Thus

$$\gamma = \phi \circ (\phi \circ \gamma) = \begin{bmatrix} -\alpha & * \\ \beta & * \end{bmatrix}$$

where $\det \gamma = -1$ and

$$\alpha = w_1 + w_3 - w_1 w_2 w_3, \quad \beta = 1 - w_2 w_3$$

Thus γ is a gluing map $\Gamma_{\alpha, \beta}$ as in (1.5) so that the boundary of this plumbing is the Seifert manifold with un-normalized Seifert invariant

$$(g = 0, b = w_0, (\alpha, \beta)).$$

Observe now that

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{w_1(1 - w_2 w_3) + w_3}{1 - w_2 w_3} = w_1 + \frac{1}{\frac{1 - w_2 w_3}{w_3}} \\ &= w_1 + \frac{1}{-w_2 + \frac{1}{w_3}} = w_1 - \frac{1}{w_2 - \frac{1}{w_3}} \end{aligned}$$

as stated in von Randow's theorem.

Observe that there are at least as many plumbing descriptions as un-normalized Seifert invariants. In fact there are more plumbing descriptions than Seifert invariants since the continuous fraction decomposition $\frac{\alpha}{\beta} = [w_1, \dots, w_k]$ is not unique. E.g. $5/3 = [2, 1] = [3, 1, 3]$. Amongst all continuous fraction decompositions of a rational number α/β , $((\alpha, \beta) = 1, \alpha > 0)$ there is a canonical one called the Hirzebrüch-Jung plumbing $\langle w_1, \dots, w_k \rangle$ uniquely determined by the requirements

$$\begin{aligned} \langle w_i, \dots, w_k \rangle &= w_i - \frac{1}{\langle w_{i+1}, \dots, w_k \rangle}, \quad \text{sign}(w_i) = \text{sign}(\langle w_i, \dots, w_k \rangle) = \text{sign}(\beta) \\ |w_i| &= \lceil |\langle w_i, \dots, w_k \rangle| \rceil, \quad \forall i = 1, \dots, k \end{aligned}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. For example

$$\frac{8}{5} = \langle 2, 3, 2 \rangle, \quad \frac{3}{-2} = \langle -2, -2 \rangle.$$

If N is a Seifert manifold with **normalized** Seifert invariant $(g, b, m; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$ then the *Hirzebrüch-Jung* plumbing is obtained from the Seifert invariant $(g, b - m, m; (\alpha_i, \beta_i - \alpha_i), i = 1, \dots, m)$ using the Hirzebrüch-Jung decompositions

$$\frac{\alpha_i}{\beta_i - \alpha_i} = \langle w_1, \dots, w_{k_i} \rangle.$$

Notice that all the weights w_i are negative since

$$\frac{\alpha_i}{\alpha_i - \beta_i} = \langle -w_1, \dots, -w_{k_i} \rangle.$$

We conclude with a convention. If the weight of the center of a star-shaped graph is $(0, w_0)$, that is the associated bundle is a disk bundle over a 2-sphere, then we say that the plumbing is *spherical* and instead of $(0, w_0)$ we will write simply w_0 .

§1.3 Seifert structures on lens spaces We now want to apply the general considerations in the previous subsections to lens spaces.

If p, q are two coprime integers, $p > 1$ we define the lens space $L(p, q)$ as the quotient of

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1\}$$

via the action of C_p given by

$$\rho_p(z_1, z_2) = (\rho_p z_1, \rho_p^q z_2). \quad (1.14)$$

Alternatively, we can describe $L(p, q)$ as a result of gluing two solid tori $D \times S^1$ along their boundaries using the gluing map $\Gamma_{q,p}$ (see [13]). This shows that we can regard a lens space as a Seifert manifold with (un-normalized) Seifert invariant $(g = 0, b = 0, (q, p))$. The plumbing discussion in the previous subsection shows that we can represent this Seifert structure as the boundary of a spherical plumbing given by a weighted starshaped graph with one branch

$$\begin{array}{c} w_0=0 \\ \bullet \end{array} - \begin{array}{c} w_1 \\ \bullet \end{array} - \dots - \begin{array}{c} w_k \\ \bullet \end{array}$$

where

$$q/p = [w_1, \dots, w_k].$$

The **normalized** Seifert invariant of the above Seifert fibration is easy to read. The rational Euler number is $\ell = -p/q$ so that $b = \lceil -\frac{p}{q} \rceil = -[p/q]$, $\alpha = q$ and β is the remainder of the division p/q . The Hirzebrüch-Jung plumbing corresponding to this normalized Seifert invariant is

$$\begin{array}{c} -w_1 \\ \bullet \end{array} - \dots - \begin{array}{c} -w_k \\ \bullet \end{array}$$

where

$$\frac{p}{q} = \langle w_1, \dots, w_k \rangle$$

We will refer to it as the *canonical Hirzebrüch-Jung plumbing* of $L(p, q)$.

In the above graph, each vertex can be regarded as the center of another star-shaped spherical graph with possible two branches. This shows that $L(p, q)$ can be regarded as a Seifert manifold in many different ways. In fact, as explained in [13] or [30], any lens space admits infinitely many Seifert structures. They all have something in common. Their bases have zero genus and they have at most two singular fibers. Moreover, as explained in [13] or [30], any Seifert fibration over S^2 with at most two singular fibers must be a Seifert fibration of a lens space. The Seifert invariants of all these Seifert fibrations are described in Sec. 4 of [13].

Perhaps, at this point it is instructive to look at the special example of $L(p, 1)$. This is the total space of the degree $-p$ circle bundle over S^2 and thus has the simple spherical plumbing description

$$\begin{array}{c} -p \\ \bullet \end{array}.$$

This is the canonical Hirzebruch-Jung plumbing of $L(p, 1)$. On the other hand

$$\frac{1}{p} = 1 - \frac{p-1}{p} = 1 - \frac{1}{\frac{p}{p-1}} = 1 - \frac{1}{1 - \frac{1}{1-p}}$$

so that it has also the plumbing description

$$\begin{array}{c} 0 \\ \bullet \end{array} - \begin{array}{c} 1 \\ \bullet \end{array} - \begin{array}{c} 1 \\ \bullet \end{array} - \begin{array}{c} 1-p \\ \bullet \end{array}.$$

If we regard one of the middle vertices as centers we obtain different Seifert fibrations structures. Since $L(p, 1) = L(p, kp+1)$ we can obtain many other Seifert structures starting from the continuous fraction decomposition of $(kp+1)/p$.

We will be interested only in those Seifert structure on a lens space such that the base is a good orbifold in the sense described in [30]. This can happen if and only if they have an (un-normalized) Seifert invariant

$$(g = 0, b = 0, (\alpha_1, \beta_1), (\alpha_2, \beta_2))$$

satisfying $\alpha_1 = \alpha_2$. These Seifert structures were determined in [26] for any lens space $L(p, q)$. There are only two of them

$$\mathbf{S}_{\pm}(p, q) = (0, 0, (\alpha_{\pm}, \beta_1^{\pm}), (\alpha_{\pm}, \beta_2^{\pm})) \quad (1.15)$$

which can be explicitly computed as follows.

- $\alpha_{\pm} = p/g.c.d.(p, q \pm 1)$
- $\beta_1^{\pm} + \beta_2^{\pm} = \mp g.c.d.(p, q \pm 1)$.
-

$$\beta_2^{\pm} \cdot \frac{q \pm 1}{g.c.d.(p, q \pm 1)} \equiv -1 \pmod{\alpha_{\pm}}.$$

We will refer to the above Seifert structures on $L(p, q)$ as the *geometric Seifert structures*. There is a more conceptual description of these structures. To present it, recall first the Hopf actions of S^1 on S^3 given by

$$h_{\pm} : (z_1, z_2) \xrightarrow{e^{i\theta}} (e^{\pm i\theta} z_1, e^{i\theta} z_2).$$

The action (1.14) of C_p commutes with these action of S^1 and thus the Hopf actions descend to two infinitesimally free S^1 -actions on the lens space $L(p, q)$. These define precisely the two geometric Seifert structures.

§1.4 Geometric structures on lens spaces All Seifert fibrations admit natural geometries, i.e. locally homogeneous Riemann metrics and their universal covers belong to a list of 6 homogeneous spaces (see[S]). In the case of lens spaces this geometry is induced from a round metric on their universal cover, S^3 . We want to describe those Seifert structures which interact in a certain way with this metric.

In Sec.1 of [21] we have described the precise meaning of this interaction (we need a (K, λ) structure in the terminology of [21]). In this case this is equivalent to asking that the Seifert structures are the quotient of the Hopf actions on S^1 modulo the action (1.14) of C_p . In other words, we must restrict to geometric Seifert structures.

Consider a lens space $N = L(p, q)$ equipped with a geometric Seifert structure with (un-normalized) invariant

$$(g = 0, b = 0, (\alpha, \beta_1), (\alpha, \beta_2)).$$

The base $\Sigma = N/S^1$ is a 2-orbifold with at most two conical points of isotropies C_{α_i} , $i = 1, 2$. Denote by $g(R)$ the metric on N induced by the round metric on the 3-sphere of radius R . The radius R will be described below. The group S^1 acts by isometries of $g(R)$ so that ζ , the infinitesimal generator of this action, is a Killing vector field. ζ is nowhere vanishing and produces an orthogonal decomposition

$$TN = \text{span}(\zeta) \oplus \text{span}(\zeta)^\perp.$$

The action of S^1 is compatible with this splitting and thus, the metric on $\text{span}(\zeta)^\perp$ induces an orbifold metric h on Σ . Now fix $R = R_0$ such that

$$\text{vol}_h(\Sigma) = \pi. \quad (1.16)$$

The radius R_0 can be explicitly determined as follows. Observe first that the volume of N is equal to

$$\text{length regular fiber} \times \text{vol}_h(\Sigma) = 2\pi^2 R_0/p$$

Since the regular fibers have length $(1/p) \times (\text{length of a great circle on } S^3(R_0)) = 2\pi R_0/p$. Hence

$$\text{vol}(N) = 2\pi^2 R_0^2/p.$$

On the other hand

$$\text{vol}(N) = \text{vol}(S^3(R_0))/p = 2\pi^2 R_0^3/p$$

from which we deduce $R_0 = 1$.

The regular fibers of N are geodesics and have the same length $2\pi/p$ so that ζ has length $1/p$. Denote by $\varphi \in \Omega^1(N)$ the $g(R_0)$ -dual of ζ . The metric $g(R_0)$ can be described as

$$g(R_0) = \varphi^2 \oplus h.$$

For $0 < r < 1$ define the anisotropic rescaling

$$g_r = (pr)^2 \varphi^2 \oplus h.$$

With respect to this metric the regular fibers have length $2\pi r$. Denote by ∇^r the Levi-Civita connection of the metric g_1 . Following [23] we define for each $t \in (0, 1]$ an isometry

$$L_t : (TN, g_{rt}) \rightarrow (TN, g_r), \zeta \mapsto t\zeta, \quad X \mapsto X \text{ if } X \perp \zeta.$$

Now set

$$\tilde{\nabla}^{r,t} := L_t \nabla^{rt} L_t^{-1}.$$

The connection $\tilde{\nabla}^{r,t}$ is compatible with g_r but it is not symmetric. In [23] we have shown that the limit $\lim_{t \rightarrow 0} \tilde{\nabla}^{r,t}$ exists and defines a connection compatible with the metric g_r . We will call this limit the *adiabatic Levi-Civita* connection of the metric g_r and we will denote it by $\tilde{\nabla}^r$.

Observe that a lens space admits two geometric Seifert structures. Arguing as above we obtain two families of Riemann metrics g_r and h_ρ . Both have positive scalar curvature (for $r, \rho \ll 1$) and there exist values $r_0, \rho_0 > 0$ (which need not be equal) such that the metrics g_{r_0} is homothetic to the metric h_{ρ_0} .

2 Froyshov invariants

§2.1 Froyshov's theorem For the reader's convenience we include here a brief description of the Froyshov invariant of a rational homology sphere. For details we refer to the original source, [8].

Suppose N is rational homology sphere equipped with a Riemann metric g . Pick a divergence free 1-form ν , thought of as a perturbation parameter for the 3-dimensional Seiberg-Witten equations $SW(g, \nu, \sigma)$ on (N, g, σ) , where σ is a $spin^c$ structure on N . Denote by \mathbb{S}_σ the bundle of complex spinors associated to σ and set $\det \sigma = \det \mathbb{S}_\sigma$. The pair (g, ν) is said to be *good* iff the following hold.

- The irreducible solutions of $SW(g, \nu, \sigma)$ are nondegenerate for all σ .
- If $\theta = (\psi = 0, A_\sigma)$ is the reducible solution of $SW(g, \nu, \sigma)$ then $\ker \mathfrak{D}_{A_\sigma} = 0$ where \mathfrak{D}_{A_σ} denotes the Dirac operator on \mathbb{S}_σ coupled with the connection A_σ on $\det \sigma$.
- If nonempty, the spaces of gradient flow lines (of the 3-dimensional Seiberg-Witten energy functional) which connect irreducible solutions form smooth moduli spaces of the correct dimension.

If the pair $(g, 0)$ is good then we will simply say the metric g is good.

For any irreducible solution α of $SW(g, \nu, \sigma)$ denote by $i(\alpha, \theta)$ the virtual dimension of the space of tunnelings (= connecting gradient flow lines) from α to θ . Define $m = m(g, \nu, \sigma)$ as the smallest nonnegative integer such that there are no tunnelings $\alpha \rightarrow \theta$ with $i(\alpha, \theta) = 2m + 1$. Now define

$$\mathbf{Froy}(N, g, \nu, \sigma) := 8m(g, \nu, \sigma) + 4\eta(\mathfrak{D}_{A_\sigma}) + \eta_{sign}(g)$$

where $\eta_{sign}(g)$ denotes the eta invariant of the odd-signature operator on N determined by the metric g . In [8] it was shown the quantity

$$\mathbf{Froy}(N, \sigma) := \inf\{\mathbf{Froy}(N, g, \nu, \sigma); (g, \nu) \text{ is good}\}.$$

Now define the Froyshov invariant of N by

$$\mathbf{Froy}(N) := \max_{\sigma} \mathbf{Froy}(N, \sigma).$$

To explain the relevance of this invariant in topology we need to introduce another, arithmetic invariant.

Consider a negative definite integer quadratic form q defined on a lattice Λ . Set $\Lambda^\sharp := \text{Hom}(\Lambda, \mathbb{Z})$. The quadratic form induces a morphism

$$I_q : \Lambda \rightarrow \Lambda^\sharp$$

and since q is nondegenerate the sublattice $I_q(\Lambda)$ has finite index δ_q in Λ^\sharp . There exists an induced *rational* quadratic form q^\sharp on Λ^\sharp by the equality

$$q^\sharp(\xi_1, \xi_2) := \frac{1}{\delta_q} \langle \xi_1, I_q^{-1}(\delta_q \xi_2) \rangle$$

where $\langle ?, ? \rangle : \Lambda^\sharp \times \Lambda \rightarrow \mathbb{Z}$ denotes the natural pairing. A vector $\xi \in \Lambda^\sharp$ is called *characteristic* if

$$\langle \xi, x \rangle \equiv q(x, x) \pmod{2}, \quad \forall x \in \Lambda.$$

We define the *Elkies invariant* of q by the equality

$$\Theta(q) := \text{rank}(q) + \max\{q^\sharp(\xi, \xi); \xi \text{ characteristic vector of } q\}$$

Note that if q is an even, negative definite form then

$$\Theta(q) = \text{rank}(q) \quad (2.1)$$

since in this case $\xi = 0$ is a characteristic vector. A result of Elkies ([6]) states that if q is a negative definite, *unimodular* quadratic form then $\Theta(q) \leq 0$ if and only if q is diagonal.

Theorem 2.1 (Froyshov, [8]) If X is a smooth, oriented, negative definite 4-manifold bounding the rational homology sphere N then

$$\Theta(q_X) \leq \mathbf{Froy}(N)$$

where q_X denotes the intersection form of X .

§2.2 Computations Consider a lens space $N = L(p, q)$ and fix a geometric Seifert fibration structure on it. The discussion in §1.4 shows that the Seifert invariants of this structure has the form

$$(g = 0, b = 0, (\alpha, \beta_1), (\alpha, \beta_2)), \quad \alpha > 0.$$

More explicitly, this is one of the Seifert structures $\mathbf{S}_\pm(p, q)$ described in (1.15).

If we regard N as the unit circle bundle determined by a line V -bundle over $\Sigma = S^2(\alpha, \alpha) = N/S^1$ then we deduce that

$$\ell := \deg L_0 = -\frac{\beta_1 + \beta_2}{\alpha} \quad (2.2)$$

and the isotropies of L_0 over the singular points are given by

$$\omega_i = (-\beta_i) \bmod \alpha_i, \quad i = 1, 2. \quad (2.3)$$

Above and in the sequel, for any $x, n \in \mathbb{Z}$ we denote by $x \bmod n$ the smallest nonnegative integer $\equiv x \bmod n$. We want to warn the reader that when $\alpha = 1$ the above Seifert structure has no singular fibers and N is a genuine smooth S^1 -bundle over S^2 of degree ℓ .

The canonical line bundle K_Σ of Σ has rational degree

$$\kappa := -\frac{2}{\alpha} \quad (2.4)$$

so that the rational Euler characteristic is

$$\chi = -\kappa = \frac{2}{\alpha}. \quad (2.5)$$

Denote by $\eta_{\text{sign}}(r)$ the eta invariant of the odd signature operator of N equipped with the deformed metric g_r (described in §1.4). $\eta_{\text{sign}}(r)$ was computed in [26]. To describe it explicitly we need to introduce the *Dedekind-Rademacher sums* defined for the first time by Hans Rademacher in [27]. More precisely, for every pair of coprime integers α, β , $\alpha > 1$ and any $x, y \in \mathbb{R}$ set

$$s(\beta, \alpha; x, y) := \sum_{r=1}^{\alpha} \left(\left(x + \beta \frac{r+y}{\alpha} \right) \right) \left(\left(\frac{r+y}{\alpha} \right) \right)$$

where for any $r \in \mathbb{R}$ we set

$$((r)) = \begin{cases} 0 & r \in \mathbb{Z} \\ \{q\} - \frac{1}{2} & r \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \quad (\{r\} := r - [r]).$$

The sums $s(\beta, \alpha) := s(\beta, \alpha; 0, 0)$ are the Dedekind sums studied in great detail in [12] and [28].

$$\eta_{sign}(r) = -\text{sign}(\ell) + \frac{2\ell}{3}(\chi r^2 - \ell^2 r^4) + \frac{\ell}{3} - 4s(\omega_1, \alpha) - 4s(\omega_2, \alpha). \quad (2.6)$$

The canonical $spin^c$ structure on the orbifold Σ (with determinant line bundle K_Σ^{-1}) determines by pullback a $spin^c$ structure on N which we denote by σ_0 . This allows us to bijectively identify the collection of $spin^c$ structures on L with the space of isomorphism classes of complex line bundles. Since $H^2(N, \mathbb{Z}) = \mathbb{Z}_p$ is pure torsion, the discussion at the end of §1.1 shows that all the line bundles on N are pullbacks of line V -bundles. Thus

$$\text{Spin}^c(N) \cong \text{Pic}^t(\Sigma)/\mathbb{Z}[L_0] \quad (2.7)$$

where $\text{Spin}^c(N)$ denotes the space of $spin^c$ structures on N . If L is a line bundle on N then the $spin^c$ structure $\sigma_0 \otimes L$ which corresponds to L has determinant line bundle

$$\det(\sigma_0 \otimes L) = L^{\otimes 2} \otimes \det \sigma_0 = L^{\otimes 2} \otimes \pi^* K_\Sigma^{-1}$$

where $\pi : N \rightarrow \Sigma$ is the natural projection. The associated bundle of complex spinors is

$$\mathbb{S}_L = L \oplus L \otimes \pi^* K_\Sigma^{-1}.$$

In [22] it was shown that, up to gauge equivalence, there is a unique flat connection on $\det \sigma_L$ which we denote by A_L . The Levi-Civita connection of g_r and A_L canonically determine a connection on \mathbb{S}_L compatible with the Clifford multiplication. Denote by \mathfrak{D}_L the associated Dirac operator and by $\eta_{dir}(L, r)$ its eta invariant.

The results of [22] show that for r sufficiently small, the unperturbed Seiberg-Witten equations corresponding to the $spin^c$ structure L have only one reducible solution. It is also nondegenerate since the scalar curvature of g_r is positive. Thus, for g_r is a good metric (in the sense of Froyshov's theorem) for $r \ll 1$ and since there is no Floer homology we deduce that

$$\mathbf{F}_r(\alpha, \beta_1, \beta_1) := \max\{4\eta_{dir}(L, r) + \eta_{sign}(r); L \in \text{Pic}^t(\Sigma)/\mathbb{Z}[L_0]\} \quad (2.8)$$

is an upper bound for the Froyshov invariant $\mathbf{Froy}(L(p, q))$.

We now show how one can use the results of [22] and [23] to provide explicit descriptions of

$$F_r(L) := 4\eta_{dir}(L, r) + \eta_{sign}(r).$$

We have to distinguish two cases.

A. $\alpha = 1$ so that N is a degree ℓ line bundle over S^2 or, as a lens space, $N = L(\ell, -1) = L(|\ell|, |\ell| - \text{sign}(\ell))$. The signature eta invariant is

$$\eta_{sign}(r) = -\text{sign}(\ell) + \frac{2\ell}{3}(\chi r^2 - \ell^2 r^4) + \frac{\ell}{3} \quad (2.9)$$

In this case there is a unique $spin$ structure on $\Sigma = S^2$ which corresponds to the unique holomorphic square root $K^{1/2}$ of K_Σ . This determines by pullback a spin structure on N and denote by σ_{spin} the $spin^c$ structure associated to it. Then

$$\sigma_{spin} = \sigma_0 \otimes \pi^* K_\Sigma^{1/2}.$$

For each integer $0 \leq k < |\ell|$ we denote by L_k the line bundle of degree k over Σ and by σ_k the $spin^c$ -structure

$$\sigma_{spin} \otimes \pi^* L_k = \sigma_0 \otimes \pi^*(K^{1/2} \otimes L_k).$$

Also let \mathfrak{D}_k denote the Dirac operator on \mathbb{S}_{σ_k} determined by the unique flat connection on $\det \sigma_k$ and denote by $\eta_{dir}(k, r)$ its eta invariant. Then

$$\text{Spin}^c(N) = \{\sigma_k; 0 \leq k < |\ell|\}$$

and

$$\mathbf{F}_r(1, \beta_1, \beta_2) = \max\{F_r(k) := 4\eta_{dir}(k, r) + \eta_{sign}(r); 0 \leq k < |\ell|\}.$$

In [22] we computed the eta invariants, not for the operator \mathfrak{D}_k , but for the adiabatic Dirac operators D_k . These are constructed using the connection on \mathbb{S}_{σ_k} induced by the adiabatic Levi-Civita connection TN and the flat connection $\det \sigma_k$. The eta invariant of \mathfrak{D}_k can be determined using variational formulæ corresponding to the affine deformation $(1-t)\mathfrak{D}_k + tD_k$. The difference $\eta_{dir}(k, r) - \eta(D_k)$ can be expressed as the sum of a continuous (transgression) term and a discontinuity contribution (spectral flow). The transgression term is expressed in the second transgression formula of [23] while the analysis in Sec.4 of [21] shows that the spectral flow contribution is zero if $r \ll 1$. We obtain the following results

- $k = 0$ (use Thm. 2.4 of [23])

$$\eta_{dir}(k, r) = \frac{\ell}{6} - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4).$$

- $0 < k < |\ell|$ (use the equality (2.22) and the second transgression formula of [23])

$$\eta_{dir}(k, r) = \frac{\ell}{6} + \frac{k^2}{\ell} - \text{sign}(\ell)k - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4).$$

Using (2.9) we deduce

$$F_r(k) = \frac{4}{\ell}k^2 - 4\text{sign}(\ell)k + \ell - \text{sign}(\ell)$$

We see that $F_r(k)$ is *independent of r* !!! Thus we

$$\mathbf{Froy}(L(\ell, -1)) \leq \max\{F_r(k); 0 \leq k < |\ell|\}. \quad (2.10)$$

We have two subcases

A₁ $\ell < 0$. The maximum above is

$$F_r([- \ell/2]).$$

Thus, when $\ell = -2m$ then

$$\mathbf{Froy}(L(2m, 1)) \leq \mathbf{F}_r(1, \beta_1, \beta_2) = 1, \quad (2m = \beta_1 + \beta_2 > 0) \quad (2.11)$$

and when $\ell = -(2m+1)$ then

$$\mathbf{Froy}(L(2m+1, 1)) \leq \mathbf{F}_r(1, \beta_1, \beta_2) = 1 - \frac{1}{2m+1}, \quad (2m+1 = \beta_1 + \beta_2 > 0). \quad (2.12)$$

A₂ $\ell > 0$. In this case the maximum is $F_r(0) = F_r(\ell) = \ell$ so that

$$\mathbf{Froy}_r(L(\ell, -1)) \leq \mathbf{F}_r(1, \beta_1, \beta_2) = \ell - 1, \quad (\ell = -(\beta_1 + \beta_2) > 0). \quad (2.13)$$

B $\alpha > 1$. The computations are similar in spirit to the ones in case **A** but obviously they are more complex due to the presence of singular fibers.

Let $L \rightarrow N$ be a line bundle over $N = S(L_0)$ and set $\sigma = \sigma_0 \otimes L \in \text{Spin}^c(N)$. To compute $\eta_{\text{dir}}(\sigma, r) := \eta_{\text{dir}}(L, r)$ we need to determine the *canonical representative* of L . This is the unique line V -bundle $\hat{L} = \hat{L}_\sigma \rightarrow \Sigma$ satisfying the conditions

$$\pi^* \hat{L} \cong L \quad (2.14)$$

$$\frac{\kappa - 2 \deg \hat{L}}{2\ell} \in [0, 1). \quad (2.15)$$

Denote by $\rho = \rho(\sigma) \in [0, 1)$ the rational number sitting in the left-hand-side of (2.15) and by $0 \leq \gamma_i = \gamma_i(\sigma) < \alpha$, $i = 1, 2$ the isotropy of the fibers of \hat{L}_σ over the singular points. Finally set

$$d(\sigma) = \frac{\kappa}{2} - \ell\rho(\sigma) = \deg \hat{L}_\sigma.$$

In Proposition 1.10 of [22] we computed the eta invariant for the adiabatic Dirac operator $D_L = D_\sigma$ defined by using the adiabatic connection on \mathbb{S}_σ and the flat connection on $\det \sigma$. To recover the eta invariant of $\mathfrak{D}_\sigma := \mathfrak{D}_L$ we use a deformation argument as in Case **A** and we deduce the following results.

- If $\rho(\sigma) = 0$ then

$$\eta_{\text{dir}}(\sigma, r) = \frac{\ell}{6} - 2 \sum_{i=1}^2 s(\omega_i, \alpha; \gamma_i(\sigma)/\alpha, 0) - \sum_{i=1}^2 \left(\left(\frac{q_i \gamma_i(\sigma)}{\alpha} \right) \right) - \frac{\ell}{6} (\chi r^2 - \ell^2 r^4) \quad (2.16)$$

where $0 \leq q_i < \alpha$ denotes the inverse of $\omega_i \bmod \alpha$.

- If $\rho(\sigma) > 0$ then

$$\begin{aligned} \eta_{\text{dir}}(\sigma, r) &= (1 - \frac{1}{\alpha})(1 - 2\rho) - \ell\rho(1 - \rho) + 2\rho + \frac{\ell}{6} \\ &- 2 \sum_{i=1}^2 s(\omega_i, \alpha; \frac{\gamma_i(\sigma) + \omega_i \rho}{\alpha}, -\rho) - \sum_{i=1}^2 \left\{ \frac{q_i \gamma_i(\sigma) + \rho}{\alpha} \right\} - \frac{\ell}{6} (\chi r^2 - \ell^2 r^4) \end{aligned} \quad (2.17)$$

where $\{x\}$ denotes the fractional part of the real number x .

The above formulæ may seem hopelessly useless. Fortunately, the Dedekind-Rademacher sums satisfy a reciprocity law (see [27]) which makes them computationally very friendly. We include here the reciprocity law for later use in this paper. To formulate it we must distinguish two cases.

- Both x and y are integers. Then

$$s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = -\frac{1}{4} + \frac{\alpha^2 + \beta^2 + 1}{12\alpha\beta}. \quad (2.18)$$

- x and/or y is not an integer. Then

$$s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = ((x)) \cdot ((y)) + \frac{\beta^2 \psi_2(y) + \psi_2(\beta y + \alpha x) + \alpha^2 \psi_2(x)}{2\alpha\beta} \quad (2.19)$$

where $\psi_2(x) := B_2(\{x\})$ and $B_2(z)$ is the second Bernoulli polynomial

$$B_2(z) = z^2 - z + \frac{1}{6}.$$

Denote by $R(\beta, \alpha; x, y)$ the right hand side in the above reciprocity identities. Note that $R(\alpha, \beta; y, x) = R(\beta, \alpha; x, y)$.

The reciprocity law, coupled with the identities

$$s(\beta, \alpha; x, y) = s(\beta - m\alpha, \alpha; x + my, y), \quad \forall m \in \mathbb{Z} \quad (2.20)$$

reduces the computation of any Dedekind-Rademacher sum to the special case $s(\beta, 1; x, y)$ which is

$$s((\beta, 1; x, y) = (\beta y + x)) \cdot ((y)) \quad (2.21)$$

Using (2.6), (2.16) and (2.17) we conclude that when $\rho(\sigma) = 0$ we have

$$\begin{aligned} F_r(\sigma) &= \ell - \text{sign}(\ell) - 8 \sum_{i=1}^2 s(\omega_i, \alpha; \gamma_i(\sigma)/\alpha, 0) \\ &\quad - 4 \sum_{i=1}^2 \left(\left(\frac{q_i \gamma_i(\sigma)}{\alpha} \right) \right) - 4 \sum_{i=1}^2 s(\omega_i, \alpha) \end{aligned} \quad (2.22)$$

and when $\rho(\sigma) > 0$ we have

$$\begin{aligned} F_r(\sigma) &= \ell - \text{sign}(\ell) + 4(1 - \frac{1}{\alpha})(1 - 2\rho) - 4\ell\rho(1 - \rho) + 8\rho \\ &\quad - 8 \sum_{i=1}^2 s(\omega_i, \alpha; \frac{\gamma_i(\sigma) + \omega_i\rho}{\alpha}, -\rho) - 4 \sum_{i=1}^2 \left\{ \frac{q_i \gamma_i(\sigma) + \rho}{\alpha} \right\} - 4 \sum_{i=1}^2 s(\omega_i, \alpha) \end{aligned} \quad (2.23)$$

Note again the r has disappeared!!!

Remark 2.2 Define more generally, for any metric g on $L(p, q)$

$$F = F_g : \text{Spin}^c(L(p, q)) \cong H^2(L(p, q), \mathbb{Z}) \cong \mathbb{Z}_p \rightarrow \mathbb{Q}, \quad \sigma \mapsto 4\eta_{dir}(\sigma, g) + \eta_{sign}(g).$$

Note that F is unchanged by rescaling the metric

$$F_{\lambda^2 g} = F_g$$

because the eta invariants are invariant under such changes. Moreover, for the metrics g_r associated to a geometric Seifert structure we have shown that F_{g_r} is independent of r . There are two geometric Seifert structures \mathbf{S}_\pm on $L(p, q)$ and correspondingly, two families of metrics g_r^\pm and thus two functions

$$F^\pm = F_{g_r^\pm} : \mathbb{Z}_p \rightarrow \mathbb{Q}.$$

As explained in §1.4 there are two radii r_\pm such that the metrics $g_{r_\pm}^\pm$ are homothetic. Thus, the two functions F^\pm must be equal. This corresponds to a collection of p identities between Dedekind-Rademacher sums. Numerical experimentations agree beautifully with this simple observation.

To put the formulæ to work we need to have a complete list of the canonical representatives of the line bundles on N . Given the isomorphism (1.12) this reduces to an elementary number theoretic problem.

According to (1.12) any line V -bundle on Σ can be uniquely represented as a collection

$$(\frac{i}{\alpha}, j \bmod \alpha, (i - j) \bmod \alpha), \quad i, j \in \mathbb{Z}.$$

Set $n = (\beta_1 + \beta_2)$ so that $\ell = -n/\alpha$. A collection as above is the canonical representative of a line bundle as above if

$$\frac{\kappa - 2i/\alpha}{-2n/\alpha} = \frac{i+1}{n} \in [0, 1).$$

Thus, when $\text{sign}(n) = -1$ we deduce that the complete list of canonical representatives is

$$\mathcal{R}_n = \{(\frac{i}{\alpha}, j \bmod \alpha, (i-j) \bmod \alpha); i = -1, -2, \dots, -|n|, 0 \leq j < \alpha\} \quad (2.24)$$

while when $\text{sign}(n) = 1$ the complete set of canonical representatives is

$$\mathcal{R}_n = \{(\frac{i}{\alpha}, j \bmod \alpha, (i-j) \bmod \alpha); i = -1, 0, \dots, |n| - 2, 0 \leq j < \alpha\}. \quad (2.25)$$

The invariant ρ of a canonical representative $\nu = (i/\alpha, j, i-j) \in \mathcal{R}$ is

$$\rho(\nu) = \frac{i+1}{n}. \quad (2.26)$$

Notice that we can identify

$$I_{n,\alpha} := \{-1, 0, \dots, |n| - 2\} \times \mathbb{Z}_\alpha \sim \mathcal{R}_n$$

via the correspondence

$$(k, j \bmod \alpha) \sim \nu \mapsto (\frac{\text{sign}(n)k - c}{\alpha}, j, -\text{sign}(n)k - c - j)$$

where $c := 1 - \text{sign}(n)$. The functions $\rho, \gamma_1, \gamma_2 : \mathcal{R} \rightarrow \mathbb{Q}$ can now be regarded as functions on $I_{n,\alpha}$. More precisely

$$\rho(k, j \bmod \alpha) = \frac{k+1}{|n|} \quad (2.27)$$

and

$$\gamma_1(k, j \bmod \alpha) = j, \quad \gamma_2(k, j \bmod \alpha) = \text{sign}(n)k - c - j. \quad (2.28)$$

Finally we can now regard F_r as a function

$$F_r = F_r(k, j) : I_{n,\alpha} \rightarrow \mathbb{Q}$$

given by (2.22), (2.23), (2.27) and (2.28). Hence

$$\mathbf{Froy}(L(p, q)) \leq \mathbf{F}_r(\alpha, \beta_1, \beta_2) = \max_{(k,j) \in I_{n,\alpha}} F_r(k, j). \quad (2.29)$$

From the proof of Proposition 7 in [8] we deduce that, since the metrics g_r have positive scalar curvature, our upper estimates are optimal. We have thus established the following result.

Theorem 2.3 If the lens space $L(p, q)$ is given a geometric Seifert structure with Seifert invariant

$$(g = 0, b = 0, (\alpha, \beta_1), (\alpha, \beta_2))$$

then

$$\mathbf{Froy}(L(p, q)) = \max_{(k,j) \in I_{n,\alpha}} F_r(k, j).$$

where the quantities $F_r(k, j)$ are described by (2.22)-(2.28) if $\alpha > 1$ and by (2.11)-(2.13) if $\alpha = 1$.

From (2.11)-(2.13) we deduce that (for $p > 0$)

$$\mathbf{Froy}(L(p, 1)) = \begin{cases} 1 & , \quad p = 2k \\ 1 - \frac{1}{p} & , \quad p = 2k + 1 \end{cases} \quad (2.30)$$

and

$$\mathbf{Froy}(L(p, p-1)) = p-1. \quad (2.31)$$

The above theorem reduces the computation of the Froyshov invariant to an elementary, albeit complex, arithmetic problem.

Example 2.4 We want to illustrate the strength and limitations of this theorem by computing the Froyshov invariants of $L(p, 2)$ p odd and > 3 . The case $L(3, 2) = L(3, -1)$ corresponds to a degree S^1 -bundle over S^2 and we have already dealt with it.

We will use the invariants $\mathbf{S}_-(p, 2)$ which seem to be computationally friendlier. We have

$$\begin{aligned} \alpha = p, \quad \ell = -\frac{1}{p}, \quad n = -1, \\ \beta_2 = p-1, \quad \beta_1 = 2-p, \quad \omega_1 = 1, \quad \omega_2 = p-2 \\ q_1 = 1, \quad q_2 = \frac{p-1}{2}. \end{aligned}$$

Thus

$$I_{n,\alpha} = \{0\} \times \mathbb{Z}_p$$

and $\rho : I_{n,\alpha} \rightarrow \mathbb{Q}$ is identically zero. Moreover, since $k = 0$ for all $(k, j) \in I_{n,\alpha}$ we have

$$\gamma_1(k, j) = j, \quad \gamma_2(k, j) = p - j - 1, \quad 0 \leq j < p.$$

Hence

$$\begin{aligned} F_r(k, j) = F_r(0, j) = -\frac{1}{p} + 1 - 8s(1, p; j/p, 0) - 8s(p-2, p; (p-j-1)/p, 0) \\ - 4((j/p)) - 4\left(\left(\frac{(p-1)(p-j-1)}{2p}\right)\right) - 4s(1, p) - 4s(p-2, p). \end{aligned}$$

We distinguish three cases.

- $0 < j < p-1$ The equality can be slightly simplified using the elementary identities

$$\begin{aligned} s(p-2, p, (p-j-1)/p, 0) &= s(p-2, p; -(j+1)/p, 0) = -s(2, p; (j+1)/p, 0) \\ s(p-2, p) &= -s(2, p) \\ -4\left(\left(\frac{j}{p}\right)\right) &= 2 - \frac{4j}{p} \end{aligned}$$

and

$$\begin{aligned} 4\left(\left(\frac{(p-1)(p-j-1)}{2p}\right)\right) &= -4\left(\left(\frac{p-1}{2} \frac{j+1}{p}\right)\right) = -4\left(\left(\frac{j+1}{2} - \frac{j+1}{2p}\right)\right) \\ &= \epsilon - \frac{2(j+1)}{p} \end{aligned}$$

where $\epsilon = 2$ if j is odd and $= 0$ if j is even. We deduce

$$F_r(j) = 1 - \frac{1}{p} + 8(s(2, p; \frac{j+1}{p}, 0) - s(1, p; \frac{j}{p}, 0))$$

$$+4(s(2, p) - s(1, p)) - \frac{4j}{p} - \frac{2(j+1)}{p} + 2 + \epsilon.$$

To compute the Dedekind-Rademacher sums we use the reciprocity formulæ (2.18)-(2.21). We deduce

$$\begin{aligned} s(2, p, (j+1)/p, 0) &= R(2, p; (j+1)/p, 0) - s(p, 2; 0, (j+1)/p) = R(2, p; (j+1)/p, 0) \\ &= \frac{5}{4p} \psi_2(0) + \frac{p}{4} \psi_2\left(\frac{j+1}{p}\right) = \frac{5}{24p} + \frac{p}{4} \left(\frac{(j+1)^2}{p^2} - \frac{j+1}{p} + \frac{1}{6} \right). \end{aligned}$$

We deduce similarly

$$s(1, p; j/p, 0) = \frac{1}{6p} + \frac{p}{2} \left(\frac{j^2}{p^2} - \frac{j}{p} + \frac{1}{6} \right).$$

After some elementary manipulations we deduce

$$8(s(2, p, (j+1)/p, 0) - s(1, p; j/p, 0)) = -\frac{2j^2}{p} + 2j \left(1 + \frac{2}{p} \right) - 2 + \frac{7}{3p} - \frac{p}{3}.$$

A similar argument leads to the equality

$$4(s(2, p) - s(1, p)) = -\frac{p}{6} + \frac{1}{6p}. \quad (2.32)$$

Together, all of the above yield after some elementary but tedious computations

$$F_r(j) = -\frac{2j^2}{p} + 2j \left(1 - \frac{1}{p} \right) - \frac{1}{2p} - \frac{p}{2} + 1 + \epsilon.$$

The above expression is quadratic in j . Its maximum on the *discrete* interval $(0, p-1)$ is achieved for j equal to one of the odd integers closest to the midpoint $(p-1)/2$. When $p = 4k+3$ there is only one such integer $(p-1)/2$ and we deduce

$$F_r\left(\frac{p-1}{2}\right) = 2$$

while when $p = 4k+1$ then $j = (p+1)/2$ is a maximum point

$$F_r\left(\frac{p+1}{2}\right) = 2 - \frac{2}{p}.$$

• $j = 0$ so that

$$\begin{aligned} F_r(0) &= 1 - \frac{1}{p} + 8(s(2, p; 1/p, 0) - s(1, p)) + 4(s(2, p) - s(1, p)) + 4 \left(\left(\frac{1}{2} - \frac{2}{p} \right) \right) \\ &\stackrel{(2.32)}{=} 1 - \frac{5}{6p} - \frac{p}{6} - \frac{8}{p} + 8(s(2, p; 1/p, 0) - s(1, p)). \end{aligned}$$

The above Dedekind-Rademacher sums can be computed using the reciprocity law and, as before, we deduce

$$8(s(2, p; 1/p, 0) - s(1, p)) + 4(s(2, p) - s(1, p)) = \frac{7}{3p} - \frac{p}{3}.$$

It is now clear that F_r cannot have a global maximum at $j = 0$. The case $j = p-1$ can be disposed of similarly and we leave it to the reader.

We have shown

$$\mathbf{Froy}(L(p, 2)) = \begin{cases} 2 & , \quad p = 4k-1 \\ 2 - \frac{2}{p} & , \quad p = 4k+1 \end{cases}. \quad (2.33)$$

The above example suggests that for large p, q the computational complexity can be overwhelming. On the other hand, these computations can be performed easily with any computer algebra system and, because of the reciprocity law, one can manipulate quite large numbers. Here are the results of some *MAPLE* experiments.

$$\mathbf{Froy}(L(p, 3)) = \begin{cases} 3 & , \quad p = 6k - 2 \\ 2 - \frac{2k}{6k-1} & , \quad p = 6k - 1 \\ 3 - \frac{3}{6k+1} & , \quad p = 6k + 1 \\ 2 - \frac{k+1}{3k+1} & , \quad p = 6k + 2 \end{cases} \quad (2.34)$$

$$\mathbf{Froy}(L(p, 4)) = \begin{cases} 4 & , \quad p = 8k - 3 \\ 2 & , \quad p = 8k - 1 \\ 4 - \frac{4}{p} & , \quad p = 8k + 1 \\ 2 - \frac{4}{p} & , \quad p = 8k + 3 \end{cases} \quad (2.35)$$

$$\mathbf{Froy}(L(p, 5)) = \begin{cases} 5 & , \quad p = 10k - 4 \\ 3 - \frac{4k-1}{10k-3} & , \quad p = 10k - 3 \\ 3 - \frac{2k}{5k-1} & , \quad p = 10k - 2 \\ 2 - \frac{2k}{10k-1} & , \quad p = 10k - 1 \\ 5 - \frac{5}{p} & , \quad p = 10k + 1 \\ 3 - \frac{2k+2}{5k+1} & , \quad p = 10k + 2 \\ 3 - \frac{4k+3}{10k+3} & , \quad p = 10k + 3 \\ 2 - \frac{k+2}{5k+2} & , \quad p = 10k + 4 \end{cases} \quad (2.36)$$

In [5] it was shown that the lens spaces $L(p^2, p+1)$ bound rational homology balls. Their Froyshov invariants are

$$\mathbf{Froy}(L(p^2, p+1)) = \begin{cases} p+1 & , \quad p \text{ even} \\ p+1 - \frac{p+1}{p^2} & , \quad p \text{ odd} \end{cases} \quad (2.37)$$

§2.3 Topological applications Let us introduce some terminology. By a *special* manifold we will understand a smooth, oriented, even, negative definite 4-manifold (with or without boundary). A *very special* manifold is a special manifold X such that $H_1(X, \mathbb{Z})$ has no 2-torsion. The following is a consequence of Froyshov's theorem 2.1 coupled with the equality (2.1).

Corollary 2.5 If N is rational homology sphere and X is a special 4-manifold bounding N then

$$b_2(X) \leq \mathbf{Froy}(N).$$

In particular, if $\mathbf{Froy}(N) < 1$ then there are no special manifolds bounding N .

Using (2.30), (2.33)-(2.36) we deduce immediately the following topological consequence.

Corollary 2.6 (a) The lens spaces $L(2k+1, 1)$ bound no special manifold.

(b) If X is a special manifold which bounds one of the spaces $L(2k, 1)$, $L(4k+1, 2)$, $L(6k-1, 3)$, $L(6k+2, 3)$, $L(8k+3, 4)$, $L(10k-1, 5)$, $L(10k+4, 5)$ must have $b_2(X) = 1$. In particular, the intersection form of X is diagonal.

Part (a) of this corollary is surprising because the lens spaces $L(2k+1, 1)$ do bound smooth, even 4-manifolds. Also, observe that $L(2k, 1)$ is the total space of the degree $-2k$ circle bundle over S^2 and bound a special manifold, the associated disk bundle D_{-2p} so, in this case, part (b) of the corollary is optimal.

Notice that

$$\begin{aligned}\frac{4k+1}{2} &= 2k - \frac{1}{-2} = [2k, -2] \\ \frac{6k-1}{6k-4} &= [2, 2, \dots, 2, -2] \\ &\quad \underbrace{\hspace{1.5cm}}_{2k-1} \\ \frac{8k+3}{4} &= [2k, -2, -2, -2]\end{aligned}$$

and

$$\frac{10k-1}{10k-6} = [2, 2, \dots, 2, -4].$$

$\underbrace{\hspace{1.5cm}}_{2k-1}$

We can now use Recipe (7.10) of [11] to compute the Rohlin invariants of the lens spaces in Corollary 2.6 (b). We have

$$\begin{aligned}\mu(L(4k+1, 2)) &= 0 \pmod{16\mathbb{Z}} \\ \mu(L(6k-1, 3)) &= -\mu(L(6k-1, 6k-4)) = -(2k-2) \pmod{16\mathbb{Z}} \\ \mu(L(8k+3, 4)) &= -2 \pmod{16\mathbb{Z}}\end{aligned}$$

and

$$\mu(L(10k-1, 5)) = -\mu(L(10k-1, 10k-6)) = (2-2k) \pmod{16\mathbb{Z}}$$

Using the definition of the Rohlin invariant we deduce that if X is a very special 4-manifold bounding one of the above lens spaces, then its signature ($= -b_2$) is congruent modulo 16 to μ . On the other hand, we know from Corollary 2.6 (b) that this signature must be -1 . We can draw the following conclusion.

Corollary 2.7 There exists no very special manifold X which bounds one of the lens spaces in the list below

$$L(4k+1, 2), L(6k-1, 3), L(8k+3, 4), L(10k-1, 5). \quad (2.38)$$

We leave the reader formulate other corollaries of the same nature. We would like to present another consequence of a slightly different nature. It relies on a recent result of Paolo Lisca.

Theorem 2.8 ([17]) Let (X, ω) be a 4-manifold with contact boundary equipped with a compatible symplectic form. Suppose that a connected component of the boundary of X admits a metric with positive scalar curvature. Then, the boundary of X is connected and X is negative (semi)definite.

It follows from the above theorem that any even, symplectic 4-manifold, with contact boundary a lens space, must be special. We have the immediate consequence.

Corollary 2.9 The lens space $L(2k+1, 1)$ cannot be the contact boundary of any symplectic manifold with even intersection form. Also, none of the spaces in the list (2.38) can be the contact boundary of a symplectic manifold with no 2-torsion in H_1 .

§2.4 Some conjectures and speculations The examples discussed so far suggest that the following arithmetic conjecture is plausible.

Conjecture 1 Suppose p, q are two coprime integers such that $p > q > 1$. Denote by \mathcal{R}_q the set of integers $0 \leq u \leq 2q$ such that $\text{g.c.d.}(u, q) = 1$. Then

- (a) $\mathbf{Froy}(L(p, q)) \leq q$.
- (b) $\mathbf{Froy}(L(2qk + 1 - q, q)) = q - 1$.
- (c) For each $u \in \mathcal{R}_q$ there exist integers A_u, B_u such that

$$\mathbf{Froy}(L(u + 2kq, q)) = \frac{A_u k + B_u}{u + 2qk}, \quad \forall k \in \mathbb{Z}_+$$

If true, part(c) of this conjecture would provide a very fast way of computing the Froyshov invariants of infinite families of lens spaces. The pair (A_u, B_u) can be viewed as defining an universal function

$$\mathcal{R}_q \rightarrow \mathbb{Z}^2, \quad u \mapsto (A_u, B_u).$$

Numerical experimentations have displayed an interesting phenomenon. First let us introduce an equivalence relation on the space of negative definite integral quadratic forms. Two such forms q_1 and q_2 are said to be equivalent if there exist two unimodular, negative definite diagonal forms δ_1, δ_2 such that

$$q_1 \oplus \delta_1 \cong q_2 \oplus \delta_2.$$

We denote this equivalence relation by \sim and the set of its equivalence classes by \mathcal{Q} . Also we denote by \mathcal{Q}_1 the subset of equivalence classes containing unimodular forms. Since the Elkies invariant of an unimodular diagonal form is zero we deduce that Θ defines a map

$$\Theta : \mathcal{Q} \rightarrow \mathbb{Q}$$

such that $\Theta(\mathcal{Q}_1) \in 8\mathbb{Z}_+$. It is believed (see [6] and [10]) that (at least for unimodular forms) Θ provides a “measure of complexity” of a negative definite quadratic form i.e. for any $k \in \mathbb{Z}_+$ the set

$$\{q \in \mathcal{Q}_1; \Theta(q) \leq 8k\}$$

is finite. (In [10] this result is proved for $k < 4$).

Thus, Froyshov result can be (loosely) interpreted as describing a topological upper bound for the complexity of the negative definite manifolds with boundary a given rational homology sphere.

On the other hand, the lens spaces, are links of complex surface singularities. More concretely, they are links of quotient singularities. These singularities can be resolved and the effect is a complex, negative definite manifold with (oriented) boundary the given lens space. There is a canonical way of performing such a resolution introduced by Hirzebrüch (see [1]) and topologically, this resolution coincides with the canonical Hirzebrüch-Jung plumbing associated to the given lens space. Denote by $HJ(p, q)$ the Hirzebrüch-Jung plumbing of $L(p, q)$, by $S_{p,q}$ the intersection form of $HJ(p, q)$ and by $\Theta_{p,q}$ the Elkies invariant of $S_{p,q}$. Froyshov’s theorem guarantees that $\Theta_{p,q} \leq \mathbf{Froy}(L(p, q))$. We claim the following stronger result is true.

Conjecture 2 $\Theta_{p,q} = \mathbf{Froy}(L(p, q))$.

Here is some evidence supporting this conjecture.

Proposition 2.10 Conjecture 2 is true for $L(p, 1)$ and $L(p, 2)$

Proof (a) $L(p, 1)$. We know that

$$HJ(p, 1) \cong \bullet^{-p}$$

with intersection form $S_{p,1} = (-p)$. When p is even, this intersection form is even so that

$$\Theta_{p,1} = 1 \stackrel{(2.30)}{=} \mathbf{Froy}(L(p, 1)).$$

When p is odd then

$$\Theta_{p,q} = 1 + \max\{\langle S_{p,1}^{-1}x, x \rangle; x \in \mathbb{Z} - \text{characteristic}\} = 1 + \max\{\langle S_{p,1}^{-1}x, x \rangle; x \in 2\mathbb{Z} + 1\}$$

$$1 - \min\{\frac{x^2}{p}; x \in 2\mathbb{Z} + 1\} = 1 - \frac{1}{p} \stackrel{(2.30)}{=} \mathbf{Froy}(L(p, 1)).$$

(b) $L(p, 2)$ Again we distinguish two cases, $p = 4k - 1$ and $p = 4k + 1$. Observe that

$$S_{4k-1,2} = \begin{bmatrix} -2k & 1 \\ 1 & -2 \end{bmatrix}$$

is even so that $\Theta_{4k-1,2} = \text{rank}(S_{4k-1,2}) = 2 \stackrel{(2.33)}{=} \mathbf{Froy}(L(p, q))$.

When $p = 4k + 1$ we have

$$S_{4k+1,2} = \begin{bmatrix} -(2k+1) & 1 \\ 1 & -2 \end{bmatrix}$$

with inverse

$$S_{4k+1,2}^{-1} = \frac{1}{4k+1} \begin{bmatrix} -2 & -1 \\ -1 & -(2k+1) \end{bmatrix}.$$

The characteristic vectors of $S_{4k+1,2}$ are determined by the congruence

$$\vec{v} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \equiv \begin{bmatrix} 2k+1 \\ 2 \end{bmatrix} \pmod{2}.$$

In particular, the vector

$$\vec{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is characteristic. We deduce

$$\begin{aligned} \mathbf{Froy}(L(4k+1, 2)) &\geq \Theta_{4k+1,2} = 2 + \max\{\langle S_{4k+1,2}^{-1}\vec{v}, \vec{v} \rangle; \vec{v} - \text{characteristic}\} \\ &\geq 2 + \langle S_{4k+1,2}^{-1}\vec{u}_0, \vec{u}_0 \rangle = 2 - \frac{2}{4k+1} \stackrel{(2.33)}{=} \mathbf{Froy}(L(4k+1, 2)). \end{aligned}$$

The proposition is proved. ■

The Hirzebruch resolution is not minimal. It can be transformed into a minimal one by blowing-down -1 -spheres. This operation changes the intersection form by a diagonal, unimodular intersection form and thus leaves the Elkies invariant unchanged. Hence in the Statement of **Conjecture 2** we can replace $\Theta_{p,q}$ with $\Theta(S_{p,q}^{min})$ where $S_{p,q}^{min}$ denotes the intersection form of the minimal resolution.

The phenomenon claimed in the above conjecture and illustrates in Proposition 2.10 is not singular. It was also remarked in [22] for a large class of Brieskorn spheres. We venture to formulate the following more general statement.

Conjecture 3 *Suppose the rational homology sphere N is the link of an isolated complex singularity. Denote by q_{min} the intersection form of the minimal resolution. Then for any negative definite smooth manifold X bounding N we have*

$$\Theta(q_X) \leq \Theta(q_{min}) = \mathbf{Froy}(N).$$

Thus, loosely speaking, the minimal resolution is the most complicated smooth 4-manifold bounding N .

3 The Casson-Walker invariant

In this section we describe a relationship between the Seiberg-Witten invariants of a lens space and its Casson-Walker invariant.

§3.1 The Seiberg-Witten invariants of a rational homology sphere We use the same notations and terminology as in §2.1. Suppose N is a rational homology. The set $Spin^c(N)$ of $spin^c$ structures on N is finite and has the same cardinality as $H := H_1(N, \mathbb{Z})$. Fix a $spin^c$ structure σ on N and a good metric g . Then the set of gauge equivalence classes of monopoles is finite. It consists of an unique nondegenerate reducible monopole $\theta = (0, A_\sigma)$ and finitely many, nondegenerate irreducible ones $\{C_k; i = 1, \dots, n\}$. Set

$$n_k = i(C_k, \theta)$$

and $F(\sigma) = 4\eta(\mathfrak{D}_{A_\sigma}) + \eta_{sign}$. The Seiberg-Witten invariant of (N, σ) is the rational number

$$\mathbf{sw}(\sigma) = \frac{1}{8}F(\sigma) - \sum_k (-1)^{n_k}. \quad (3.1)$$

In [3] and [15] it was proved that $\mathbf{sw}(\sigma)$ is independent of the choice of the good metric g and

$$\mathbf{sw}(\sigma) \in \frac{1}{8h_1}\mathbb{Z}$$

where $h_1 = \#H_1(N, \mathbb{Z})$. Observe that $\mathbf{sw}(\sigma) = \mathbf{sw}(\bar{\sigma})$ where $\sigma \mapsto \bar{\sigma}$ is the natural involution on $Spin^c(N)$. Set

$$\mathbf{sw}(N) := \sum_{\sigma} \mathbf{sw}(\sigma). \quad (3.2)$$

If N is a lens space $L(p, q)$ then, as explained in §2.2, a geometric Seifert structure on N determines a $spin^c$ -structure σ_0 on N . Will work with the geometric Seifert structure determined by $\alpha = p/g.c.d.(p, q - 1)$ and we set

$$\mathbf{SW}_{p,q} = \sum_{j=0}^{p-1} \mathbf{sw}(\sigma_0 \cdot t) t^j$$

where t is a generator of the cyclic group \mathbb{Z}_p . Observe that

$$\mathbf{sw}(L(p, q)) = \mathbf{aug}(\mathbf{SW}_{p,q}).$$

The *Casson-Walker* invariant of N is defined in [14] and [33]. It is a rational number $CW(N)$ uniquely determined by certain Dehn surgery properties.

We will work with C. Lescop's normalization used in [14]. It is related to K. Walker's normalization used in [33] by the equality ([14, Property T5.0, p.76]

$$CW(N)_{Lescop} = \frac{h_1}{2} CW(N)_{Walker}.$$

Remark 3.1 The Casson-Walker invariant with C. Lescop's normalization differs by a sign from the conventions for the Casson invariant used in [7] and [24]. In these references the Casson invariant is normalized so that for the Brieskorn homology sphere $\Sigma(a, b, c)$ we have

$$CW(\Sigma(a, b, c)) = \frac{1}{8}\sigma(a, b, c)$$

where $\sigma(a, b, c)$ denotes the signature of the Milnor fiber associated to $\Sigma(a, b, c)$. In particular for the Poincaré sphere $\Sigma(2, 3, 5)$ we have $\sigma(2, 3, 5) = \text{sign}(-E_8) = -8$ so that the above formula gives the value -1 for the Casson invariant. On the other hand using C. Lescop's formula [14, Prop. 6.1.1, To.0, p.97] we obtain the value 1 for the Casson-Walker invariant. This explains the sign difference between the definition of **sw** in (3.1) and (3.2) and the definition in [24].

The Casson-Walker invariant of the lens space can be expressed in terms of the Dedekind sums. More precisely we have the equality (see [2], [33])

$$CW(L(p, q)) = -\frac{p}{2}s(q, p). \quad (3.3)$$

We can now state the main result of this section.

Theorem 3.2

$$\mathbf{sw}(L(p, q)) = CW(L(p, q)).$$

§3.2 Seiberg-Witten \Rightarrow Casson-Walker Our proof of Theorem 3.2 is arithmetic in nature and relies on the computations in §2.2.

We will work with the same metric as in §2.2 and, since it has positive scalar curvature we deduce there are no irreducible monopoles, the unique reducible is also nondegenerate and thus

$$\mathbf{sw}(L(p, q), \sigma) = \frac{1}{8}F_{p,q}(\sigma), \quad \forall \sigma \in \text{Spin}^c(L(p, q)).$$

To proceed further we need to organize the computational facts established in §2.2 in a form suitable to our current purposes.

Set $n = \text{g.c.d.}(p, q - 1)$, $\alpha = p/n$

$$\beta_2 \cdot \frac{q-1}{n} \equiv -1 \pmod{\alpha}, \quad \beta_1 = n - \beta_2$$

$$\omega_i = -\beta_i, \quad q_i \omega_i \equiv 1 \pmod{\alpha} \quad \forall i = 1, 2.$$

The rational Euler degree of $L(p, q)$ equipped with the above geometric Seifert structure is

$$\ell = -\frac{n}{\alpha} = -\frac{n^2}{p}.$$

For each positive integer m set $I_m := \{0, 1, \dots, m-1\}$ and $I_m^* = \{1, \dots, m-1\}$. The set $\text{Spin}^c(L(p, q))$ can be identified with $I_n \times I_\alpha$ and we have several functions of interest

$$\rho : I_n \times I_\alpha \rightarrow \mathbb{Q}, \quad \rho(k, j) = \frac{k}{n}$$

$$\gamma_1, \gamma_2 : I_n \times I_\alpha \rightarrow \mathbb{Z}, \quad \gamma_1(k, j) = j, \quad \gamma_2(k, j) = k - 1 - j.$$

The function $F_{p,q}(\sigma)$ can be regarded as a function $F : I_n \times I_\alpha \rightarrow \mathbb{Q}$. It is explicitly described by

$$F(k, j) = \ell + 1 - 4\ell\rho(1 - \rho) + 8\rho - 4 \sum_{i=1}^2 s(\omega_i, \alpha) - 8 \sum_{i=1}^2 s(\omega_i, \alpha, \frac{\gamma_i + \omega_i \rho}{\alpha}, -\rho) \\ + 4 \begin{cases} - \sum_{i=1}^2 \left(\left(\frac{q_i \gamma_i}{\alpha} \right) \right) & \text{if } \rho = 0 \\ (1 - \frac{1}{\alpha})(1 - 2\rho) - \sum_{i=1}^2 \left\{ \frac{q_i \gamma_i + \rho}{\alpha} \right\} & \text{if } \rho \neq 0 \end{cases} \quad (3.4)$$

We have to prove

$$\sum_{k \in I_n} \sum_{j \in I_\alpha} F(k, j) = -4ps(q, p). \quad (3.5)$$

The proof of (3.5) relies on two identities. The first one was proved by M. Ouyang, [26, p.652]. More precisely, we have

$$\sum_{i=1}^2 s(\omega_i, \alpha) = s(q, p) - \frac{1}{6p} - \frac{n^2}{12p} + \frac{1}{4}. \quad (3.6)$$

The second one is central in the theory of Dedekind sums and has the form

$$\sum_{\mu \in I_m} \left(\left(\frac{\mu + w}{m} \right) \right) = ((w)), \quad \forall m \in \mathbb{Z}_+, w \in \mathbb{R}. \quad (3.7)$$

For a proof we refer to [12].

Summing (3.6) over $(k, j) \in I_n \times I_\alpha$ and using the equality $p = n\alpha$ we deduce

$$4 \sum_{k \in I_n} \sum_{j \in I_\alpha} s(\omega_i, \alpha) = 4ps(q, p) - \frac{2}{3} - \frac{n^2}{3} + p. \quad (3.8)$$

We now proceed to sum over $(k, j) \in I_n \times I_\alpha$ all the terms entering into the definition of $F(k, j)$.

$$\sum_{k \in I_n} \sum_{j \in I_\alpha} (\ell + 1) = -n^2 + p. \quad (3.9)$$

$$8 \sum_{k \in I_n} \sum_{j \in I_\alpha} \rho = 8 \sum_{j \in I_\alpha} \sum_{k \in I_n} \frac{k}{n} = \frac{8\alpha}{n} \frac{n(n-1)}{2} = 4(p - \alpha). \quad (3.10)$$

$$4\ell \sum_{k \in I_n} \sum_{j \in I_\alpha} \rho(1 - \rho) = -\frac{4n}{\alpha} \sum_{j \in I_\alpha} \sum_{k \in I_n} \frac{k(n-k)}{n^2} = -\frac{4}{n} \sum_{k \in I_n} k(n-k)$$

$$(\sum_{k \in I_n} k^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}) \\ = -\frac{4}{n} \left(\frac{n^3}{2} - \frac{n^2}{2} - \frac{n^3}{3} + \frac{n^2}{2} - \frac{n}{6} \right) = -\frac{2}{3}n^2 + \frac{2}{3}. \quad (3.11)$$

$$\sum_{k \in I_n} \sum_{j \in I_\alpha} S(\omega_i, \alpha, \frac{\gamma_i + \omega_i \rho}{\alpha}, -\rho) = \sum_{k \in I_n} \sum_{j \in I_\alpha} \sum_{\mu \in I_\alpha} \left(\left(\frac{\mu - \rho}{\alpha} \right) \right) \left(\left(\frac{(\omega_i(\mu - \rho) + \gamma_i + \omega_i \rho)}{\alpha} \right) \right) \\ \sum_{\mu \in I_\alpha} \left(\left(\frac{\mu - \rho}{\alpha} \right) \right) \sum_{k \in I_n} \sum_{j \in I_\alpha} \left(\left(\frac{\gamma_i(k, j) + \omega_i \mu}{\alpha} \right) \right) = \sum_{\mu \in I_\alpha} \left(\left(\frac{\mu - \rho}{\alpha} \right) \right) \sum_{k \in I_n} \sum_{r \in I_\alpha} \left(\left(\frac{r + \omega_i \mu}{\alpha} \right) \right).$$

According to (3.7), the last sum (over r) is equal to $((\omega_i \mu)) = 0$. Hence

$$\sum_{k \in I_n} \sum_{j \in I_\alpha} S(\omega_i, \alpha, \frac{\gamma_i + \omega_i \rho}{\alpha}, -\rho) = 0. \quad (3.12)$$

Using (3.7) again we deduce

$$\sum_{k \in I_n^*} \sum_{j \in I_\alpha} \left(\left(\frac{q_i \gamma_i(k, j)}{\alpha} \right) \right) = \sum_{k \in I_n^*} \sum_{r \in I_\alpha} \left(\left(\frac{r}{\alpha} \right) \right) \stackrel{(3.7)}{=} 0. \quad (3.13)$$

Observe that since $1 - 2\rho(k) = -(1 - 2\rho(n - k))$ we have

$$(1 - \frac{1}{\alpha}) \sum_{k \in I_n^*} \sum_{j \in I_\alpha} (1 - 2\rho) = 0. \quad (3.14)$$

Finally we have

$$\begin{aligned} \sum_{k \in I_n^*} \sum_{j \in I_\alpha} \left\{ \frac{q_i \gamma_i(k, j) + \rho(k)}{\alpha} \right\} &= \sum_{k \in I_n} \sum_{j \in I_\alpha} \left\{ \frac{q_i \gamma_i(k, j) + \rho(k)}{\alpha} \right\} - \sum_{j \in I_\alpha} \left\{ \frac{q_i \gamma_i(0, j)}{\alpha} \right\} \\ &= \sum_{k \in I_n} \sum_{j \in I_\alpha} \left\{ \frac{nq_i \gamma_i(k, j) + k}{p} \right\} - \sum_{j \in I_\alpha} \left\{ \frac{q_i \gamma_i(0, j)}{\alpha} \right\}. \end{aligned}$$

Now observe that as k covers I_n and j covers I_α the quantity $(nq_i \gamma_i(k, j) + k \bmod p)$ covers I_p while $q_i \gamma_i(0, j)$ covers I_α . Hence

$$\sum_{k \in I_n} \sum_{j \in I_\alpha} \left\{ \frac{nq_i \gamma_i(k, j) + k}{p} \right\} = \sum_{r \in I_p} \left\{ \frac{r}{p} \right\} = \frac{p-1}{2}$$

and

$$\sum_{j \in I_\alpha} \left\{ \frac{q_i \gamma_i(0, j)}{\alpha} \right\} = \sum_{r \in I_\alpha} \left\{ \frac{r}{\alpha} \right\} = \frac{\alpha-1}{2}.$$

We conclude that

$$4 \sum_{k \in I_n^*} \sum_{j \in I_\alpha} \sum_{i=1}^2 \left\{ \frac{q_i \gamma_i(k, j) + \rho(k)}{\alpha} \right\} = 4(p - \alpha). \quad (3.15)$$

The identity (3.5) now follows from (3.8)-(3.15). Theorem 3.2 is proved. ■

Taking into account Theorem 3.2 and the results of [4] and [16] it is very tempting to formulate the following conjecture.

Conjecture 4 *For every rational homology sphere N we have*

$$\mathbf{sw}(N) = CW(N)$$

§3.3 Seiberg-Witten invariants and Milnor torsion Consider the invariant $T_{p,q}$ of the lens space $L(p, q)$ described in the introduction. The goal of this section is to prove the following result.

Proposition 3.3 If $\gcd(p, q - 1) = 1$ then

$$T_{p,q}(1 - t)(1 - t^q) \sim \hat{\mathbf{1}} \quad (3.16)$$

i.e. $T_{p,q} \sim \tau_{p,q}$.

Proof For a while we will not rely on the assumption $\gcd(p, q - 1) = 1$. We will continue to use the notations in the previous subsection. Thus $n = \gcd(p, q - 1)$.

As explained in §2.2, each $(k, j) \in I_n \times I_\alpha \cong I_{n,\alpha}$ defines a line bundle on L_j on $L(p, q)$ and thus, via the first Chern class an element $e(k, j) = c_1(L_{k,j}) \in H^2(L(p, q), \mathbb{Z} \cong \mathbb{Z}_p)$. Moreover, the correspondence

$$e : I_n \times I_\alpha \rightarrow \mathbb{Z}_p, \quad (k, j) \mapsto e(k, j)$$

is a bijection.

Lemma 3.4 There exists an isomorphism of abelian groups $H^2(L(p, q), \mathbb{Z}) \rightarrow \mathbb{Z}_p$ such that

$$e(k, j) = q(k - 1) - (q - 1)j \bmod p.$$

Proof of the lemma $H^2(L(p, q), \mathbb{Z})$ is torsion so according to the results in §1.1 it can be described in terms of the chosen geometric Seifert structure as follows.

Consider map $\mathbb{Q} \oplus \mathbb{Z}_\alpha \oplus \mathbb{Z}_\alpha \rightarrow \mathbb{Q}/\mathbb{Z}$

$$(d, \gamma_1, \gamma_2) \mapsto d - \frac{\gamma_1 + \gamma_2}{\alpha}$$

and the element

$$L_0 = (-n, \omega_1, \omega_2) \in \ker \delta.$$

Recall that L_0 describes a line V -bundle over an genus 0 orbifold whose associated circle bundle coincides with the lens space equipped with the chosen Seifert structure. Then

$$H^2(L(p, q), \mathbb{Z}) \cong \ker \delta / \mathbb{Z}[L_0].$$

Now observe that $\ker \delta / \mathbb{Z}[L_0]$ has the presentation

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \rightarrow \ker \delta / \mathbb{Z}[L_0] \rightarrow 0$$

where

$$A = \begin{bmatrix} -n & 0 \\ \omega_1 & \alpha \end{bmatrix}.$$

We let the reader verify that

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ q & 1 - q \end{bmatrix} \cdot A \cdot \begin{bmatrix} y & -\alpha \\ -x & -\omega_2 \end{bmatrix} \quad (3.17)$$

where

$$y = -(q-1)/n \text{ and } x = -\frac{\omega_2 y + 1}{\alpha}.$$

This shows that indeed

$$\ker \delta / \mathbb{Z}[L_0] \cong \mathbb{Z}_p.$$

To each pair $(k, j) \in I_n \times I_\alpha$ it corresponds the line bundle $L_{k,j}$ with Seifert data $(k-1, j, k-1-j) \in \ker \delta$. Its first Chern class is the image of the vector $\vec{v} = (k-1, j) \in \mathbb{Z}^2$ in the quotient $\mathbb{Z}^2 / A\mathbb{Z}^2$. Using the equality (3.17) we deduce that this image is $(y_2 \bmod p)$ where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ q & 1-q \end{bmatrix} \cdot \begin{bmatrix} k-1 \\ j \end{bmatrix}.$$

This establishes the assertion in the lemma. **q.e.d.**

Denote by $c : \mathbb{Z}_p \rightarrow I_n \times I_\alpha$ the inverse of the map e described in the above lemma.

Lemma 3.5 We have the following equalities.

(i) If $n = 1$ then $\alpha = p$ and

$$c(m) = (0, -\omega_2 m + \omega_1 \bmod p).$$

(ii) If $n \geq 1$ then

$$c(-1) = c(p-1) = (0, \alpha-1) \text{ and } c(-m) = c(p-m) = (r, (-m - s\omega_1) \bmod \alpha), \quad \forall m \in I_p$$

where $r \in I_n$ and $s \in \mathbb{Z}$ are such that $ns = (m-1) + r$ so that

$$r = -(m-1) \bmod n \text{ and } s = \left\lceil \frac{m-1}{n} \right\rceil$$

where $\lceil x \rceil$ is the smallest integer $\geq x$.

Proof We prove only part (i). The second part is left to the reader.

Observe that when $n = 1$ we have $I_n \times I_\alpha = \{0\} \times I_\alpha$. Thus we can write $c(m) = (0, j)$ where

$$m = -q - (q-1)j \bmod p$$

Since $\omega_2 = (q-1)^{-1} \bmod p$ we have the following mod p equalities

$$\omega_2 m = -q\omega_2 - j = -(q-1+1)\omega_2 - j = -\omega_2 - 1 - j.$$

The equality in (i) now follows from $\omega_1 + \omega_2 = -n = -1$. **q.e.d**

Now we can write

$$\mathbf{SW}_{p,q} = \frac{1}{8} \sum_{m \in I_p} F(c(m)) t^m.$$

Since $\Sigma \cdot (1-t) = 0$ in $\mathbb{Q}[\mathbb{Z}_p]$ the equality (3.16) is equivalent to

$$\mathbf{SW}_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}}.$$

We will prove a slightly stronger statement namely

$$\mathbf{SW}_{p,q}(1-t)(1-t^q) = \hat{\mathbf{1}}. \tag{3.18}$$

Let us introduce the polynomial

$$f(t) = \sum_{j \in I_p} \left(\left(\frac{j}{p} \right) \right) t^j \in \mathbb{Q}[\mathbb{Z}_p].$$

A simple computation shows that

$$f(t^{-1}) = -f(t)$$

and for all m coprime with p we have

$$\left(\frac{1}{2} - f(t^m) \right) (1 - t^m) = \hat{\mathbf{1}} \text{ in } \mathbb{Q}[\mathbb{Z}_p] \quad (3.19)$$

We want to express $\mathbf{SW}_{p,q}$ as a linear combinations of polynomials of the form $t^a f(t^a)$, $t^a f(t^a) f(t^b)$ and Σ . Observe first that since $n = 1$, in the equality (3.4) of §3.2 we always have $\rho = 0$. Thus for all $(k, j) \in I_n \times I_\alpha$ we have

$$\begin{aligned} F(k, j) &= \ell + 1 - 4 \sum_{i=1}^2 s(\omega_i, \alpha) \\ &\quad - 8 \sum_{i=1}^2 s(\omega_i, \alpha, \gamma_i(k, j)/\alpha, 0) - 4 \sum_{i=1}^2 \left(\left(\frac{q_i \gamma_i(k, j)}{\alpha} \right) \right). \end{aligned}$$

Observe two things.

- Since $n = 1$ we always have $k = 0 \in I_1 = \{0\}$ so that we can write $\gamma_1(j)$ instead of $\gamma_1(k, j)$.
- The first term in the definition of $F(k, j)$ is independent of (k, j) . Thus its contribution to $\mathbf{SW}_{p,q}$ will be of the form $\text{const.} \Sigma$ which is cancelled upon multiplication by $(1 - t)$. Thus when computing $\mathbf{SW}_{p,q}(1 - t)(1 - t^q)$ we can neglect this first term.

For $i = 1, 2$ define

$$A_i = -8 \sum_{m \in I_p} s\left(\omega_i, \alpha, \frac{\gamma_i(c(m))}{\alpha}, 0\right) t^m, \quad B_i = \sum_{m \in I_p} \left(\left(\frac{q_i \gamma_i(c(m))}{\alpha} \right) \right) t^m$$

where according to §3.2 we have

$$\gamma_1(j) = j, \quad \gamma_2(j) = -1 - j$$

so that according to Lemma 3.5 we have

$$\gamma_1(c(m)) = -\omega_2 m + \omega_1, \quad \gamma_2(c(m)) = \omega_2 m - \omega_1 - 1 = \omega_2(m + 1).$$

Observe that since $q_2 \omega_2 = 1 \pmod{p}$ and $\omega_2(q - 1) = 1 \pmod{p}$ we have

$$q_2 = (q - 1) \pmod{p}.$$

Lemma 3.6

$$B_1 = -t^{-q} f(t^{-q}) \quad (3.20)$$

$$B_2 = -t^{-1} f(t^{-1}) \quad (3.21)$$

$$A_1 = -t^{-q} f(t^{-q}) f(t^{q-1}) \quad (3.22)$$

$$A_2 = t^{-1} f(t^{-1}) f(t^{q-1}) \quad (3.23)$$

Proof For any $(m, p+1)$ we will denote by $1/m$ the inverse of $m \bmod p$.

$$B_1 = - \sum_{m \in I_m} \left(\left(\frac{q_1(\omega_2 m - \omega_1)}{p} \right) \right) t^m$$

$$(\mu := q_1 \omega_2 - q_1 \omega_1 = q_1 \omega_2 m - 1, m = \frac{\omega_1}{\omega_2}(\mu + 1))$$

$$= -t^{\omega_1/\omega_2} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\omega_1 \mu / \omega_2} = -t^{\omega_1/\omega_2} f(t^{\omega_1/\omega_2}).$$

Now observe that $1/\omega_2 = q_2 = q - 1$ and $\omega_1 = -1 - \omega_2$ so that $\omega_1/\omega_2 = -q$. This proves (3.20).

$$B_2 = \sum_{m \in I_m} \left(\left(\frac{q_2 \omega_2(m+1)}{p} \right) \right) t^m = \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\mu-1} = t^{-1} f(t) = -t^{-1} f(t^{-1}).$$

This proves (3.21).

$$A_1 = \sum_{m \in I_p} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \left(\left(\frac{\omega_1 \mu - \omega_2 m + \omega_1}{p} \right) \right) t^m = \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \sum_{m \in I_p} \left(\left(\frac{\omega_1 \mu - \omega_2 m + \omega_1}{p} \right) \right) t^m$$

$$(r = \omega_1 \mu - \omega_2 m + \omega_1, m = -r/\omega_2 + \omega_1(\mu + 1)/\omega_2)$$

$$= t^{\omega_1/\omega_2} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\omega_1 \mu / \omega_2} \sum_{r \in I_p} \left(\left(\frac{r}{p} \right) \right) t^{-r/\omega_2} = t^{\omega_1/\omega_2} f(t^{\omega_1/\omega_2}) f(t^{-1/\omega_2})$$

$$= t^{-q} f(t^{-q}) f(t^{-(q-1)}) = -t^{-q} f(t^{-q}) f(t^{q-1}).$$

This proves (3.22). Finally, we have

$$A_2 = \sum_{m \in I_p} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \left(\left(\frac{\omega_2 \mu + \omega_2 m + \omega_2}{p} \right) \right) t^m = \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \sum_{m \in I_p} \left(\left(\frac{\omega_2 \mu + \omega_2 m + \omega_2}{p} \right) \right) t^m$$

$$(r = \omega_2(m + \mu + 1), m = r/\omega_2 - \mu - 1)$$

$$= t^{-1} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{-\mu} \sum_{r \in I_p} \left(\left(\frac{r}{p} \right) \right) t^{r/\omega_2} = t^{-1} f(t^{-1}) f(t^{q-1})$$

This proves (3.23). **q.e.d.**

We can now finish the proof of Proposition 3.3. Using Lemma 3.6 we deduce

$$\begin{aligned} 8\mathbf{SW}_{p,q}(1-t)(1-t^q) &= (-8A_1 - 8A_2 - 4B_1 - B_2 + \text{const.}\Sigma)(1-t)(1-t^q) \\ &= -4(2A_1 + 2A_2 + B_1 + B_2)(1-t)(1-t^q) \\ &= -4 \left\{ -t^{-q} f(t^{-q})(1 + 2f(t^{q-1})) - t^{-1} f(t^{-1})(1 - 2f(t^{q-1})) \right\} (1-t)(1-t^q) \\ &= -8 \left\{ -t^{-q} f(t^{-q}) \left(\frac{1}{2} - f(t^{-(q-1)}) \right) - t^{-1} f(t^{-1}) \left(\frac{1}{2} - f(t^{q-1}) \right) \right\} (1-t)(1-t^q) \\ &\stackrel{(3.19)}{=} 8 \left\{ t^{-q} f(t^{-q}) \cdot \frac{\hat{\mathbf{1}}}{1-t^{1-q}} + t^{-1} f(t^{-1}) \cdot \frac{\hat{\mathbf{1}}}{1-t^{q-1}} \right\} (1-t)(1-t^q) \end{aligned}$$

$$\begin{aligned}
&= 8 \left\{ t^{-1} f(t^{-q}) \cdot \frac{\hat{\mathbf{1}}}{t^{q-1} - 1} + t^{-1} f(t^{-1}) \cdot \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} \right\} (1-t)(1-t^q) \\
&= 8t^{-1} \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} (f(t^{-1}) - f(t^{-q})) (1-t)(1-t^q) \\
&= 8t^{-1} \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} (f(t^q) - f(t)) (1-t)(1-t^q) \\
&\stackrel{(3.19)}{=} 8t^{-1} \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} \left(\frac{\hat{\mathbf{1}}}{1-t} - \frac{\hat{\mathbf{1}}}{1-t^q} \right) (1-t)(1-t^q) \\
&= 8t^{-1} \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} \{ (1-t^q) - (1-t) \} = 8t^{-1} \frac{\hat{\mathbf{1}}}{1 - t^{q-1}} (t - t^q) = 8 \cdot \hat{\mathbf{1}}.
\end{aligned}$$

The proof of Proposition 3.3 is now complete. ■

The restriction $g.c.d.(p, q-1) = 1$ in Proposition 3.3 can be dropped but we will present the details elsewhere. Instead, we have included below an explicit description of $T_{p,q}$ for all $1 < q < p \leq 10$. These computations confirm the validity of (3.16) even if $g.c.d.(p, q-1) > 1$.

Proposition 3.3 confirms a hypothesis formulated in [32]. We formulate the following conjecture.

Conjecture 5 *For any rational homology sphere N the augmentation-free part of the Seiberg-Witten invariant coincides with the refined torsion of Turaev, [32].*

Numerical experiments

Below we let the reader verify the elementary identity (3.16) in each case.

- L(2,q)

$$T_{2,1} \sim -\frac{1}{8}t + \frac{1}{8}$$

- L(3,q)

$$T_{3,1} \sim \frac{1}{9}t^2 - \frac{2}{9}t + \frac{1}{9}$$

$$T_{3,2} \sim -\frac{1}{9}t^2 + \frac{2}{9}t - \frac{1}{9}$$

- L(4,q)

$$T_{4,1} \sim -\frac{5}{16}t^3 + \frac{1}{16}t^2 + \frac{3}{16}t + \frac{1}{16}$$

$$T_{4,3} \sim -\frac{5}{16}t^3 + \frac{1}{16}t^2 + \frac{3}{16}t + \frac{1}{16}$$

- L(5,q)

$$T_{5,1} \sim \frac{1}{5}t^4 - \frac{2}{5}t^2 + \frac{1}{5}$$

$$T_{5,2} \sim -\frac{1}{5}t^4 + \frac{1}{5}t^3 + \frac{1}{5}t - \frac{1}{5}$$

$$T_{5,3} \sim -\frac{1}{5}t^4 + \frac{1}{5}t^3 + \frac{1}{5}t - \frac{1}{5}$$

$$T_{5,4} \sim -\frac{1}{5}t^3 + \frac{2}{5}t^2 - \frac{1}{5}t$$

• L(6,q)

$$T_{6,1} \sim -\frac{35}{72}t^5 - \frac{5}{72}t^4 + \frac{13}{72}t^3 + \frac{19}{72}t^2 + \frac{13}{72}t - \frac{5}{72}$$

$$T_{6,5} \sim \frac{35}{72}t^5 + \frac{5}{72}t^4 - \frac{13}{72}t^3 - \frac{19}{72}t^2 - \frac{13}{72}t + \frac{5}{72}$$

• L(7,q)

$$T_{7,1} \sim \frac{2}{7}t^6 + \frac{1}{7}t^5 - \frac{1}{7}t^4 - \frac{4}{7}t^3 - \frac{1}{7}t^2 + \frac{1}{7}t + \frac{2}{7}$$

$$T_{7,2} \sim -\frac{2}{7}t^6 + \frac{1}{7}t^5 + \frac{2}{7}t^3 + \frac{1}{7}t - \frac{2}{7}$$

$$T_{7,3} \sim -\frac{1}{7}t^6 + \frac{2}{7}t^4 - \frac{2}{7}t^3 + \frac{2}{7}t^2 - \frac{1}{7}$$

$$T_{7,4} \sim -\frac{2}{7}t^6 + \frac{1}{7}t^5 + \frac{2}{7}t^3 + \frac{1}{7}t - \frac{2}{7}$$

$$T_{7,5} \sim -\frac{1}{7}t^6 + \frac{2}{7}t^4 - \frac{2}{7}t^3 + \frac{2}{7}t^2 - \frac{1}{7}$$

$$T_{7,6} \sim \frac{1}{7}t^6 - \frac{2}{7}t^5 - \frac{1}{7}t^4 + \frac{4}{7}t^3 - \frac{1}{7}t^2 - \frac{2}{7}t + \frac{1}{7}$$

• L(8,q)

$$T_{8,1} \sim -\frac{21}{32}t^7 - \frac{7}{32}t^6 + \frac{3}{32}t^5 + \frac{9}{32}t^4 + \frac{11}{32}t^3 + \frac{9}{32}t^2 + \frac{3}{32}t - \frac{7}{32}$$

$$T_{8,3} \sim -\frac{9}{32}t^7 - \frac{3}{32}t^6 - \frac{9}{32}t^5 + \frac{5}{32}t^4 + \frac{7}{32}t^3 - \frac{3}{32}t^2 + \frac{7}{32}t + \frac{5}{32}$$

$$T_{8,5} \sim -\frac{5}{32}t^7 - \frac{7}{32}t^6 + \frac{3}{32}t^5 - \frac{7}{32}t^4 - \frac{5}{32}t^3 + \frac{9}{32}t^2 + \frac{3}{32}t + \frac{9}{32}$$

$$T_{8,7} \sim \frac{21}{32}t^7 - \frac{9}{32}t^6 - \frac{3}{32}t^5 + \frac{7}{32}t^4 - \frac{11}{32}t^3 + \frac{7}{32}t^2 - \frac{3}{32}t - \frac{9}{32}$$

• L(9,q)

$$T_{9,1} \sim \frac{10}{27}t^8 + \frac{7}{27}t^7 + \frac{1}{27}t^6 - \frac{8}{27}t^5 - \frac{20}{27}t^4 - \frac{8}{27}t^3 + \frac{1}{27}t^2 + \frac{7}{27}t + \frac{10}{27}$$

$$T_{9,2} \sim -\frac{10}{27}t^8 + \frac{2}{27}t^7 - \frac{1}{27}t^6 + \frac{8}{27}t^5 + \frac{2}{27}t^4 + \frac{8}{27}t^3 - \frac{1}{27}t^2 + \frac{2}{27}t - \frac{10}{27}$$

$$T_{9,4} \sim \frac{1}{27}t^8 - \frac{2}{27}t^7 - \frac{8}{27}t^6 + \frac{10}{27}t^5 - \frac{2}{27}t^4 + \frac{10}{27}t^3 - \frac{8}{27}t^2 - \frac{2}{27}t + \frac{1}{27}$$

$$T_{9,5} \sim -\frac{10}{27}t^8 + \frac{2}{27}t^7 - \frac{1}{27}t^6 + \frac{8}{27}t^5 + \frac{2}{27}t^4 + \frac{8}{27}t^3 - \frac{1}{27}t^2 + \frac{2}{27}t - \frac{10}{27}$$

$$T_{9,7} \sim -\frac{8}{27}t^8 - \frac{2}{27}t^7 + \frac{10}{27}t^6 + \frac{1}{27}t^5 - \frac{2}{27}t^4 + \frac{1}{27}t^3 + \frac{10}{27}t^2 - \frac{2}{27}t - \frac{8}{27}$$

$$T_{9,8} \sim \frac{8}{27}t^8 - \frac{7}{27}t^7 - \frac{10}{27}t^6 - \frac{1}{27}t^5 + \frac{20}{27}t^4 - \frac{1}{27}t^3 - \frac{10}{27}t^2 - \frac{7}{27}t + \frac{8}{27}$$

• L(10,q)

$$T_{10,1} \sim -\frac{33}{40}t^9 - \frac{3}{8}t^8 - \frac{1}{40}t^7 + \frac{9}{40}t^6 + \frac{3}{8}t^5 + \frac{17}{40}t^4 + \frac{3}{8}t^3 + \frac{9}{40}t^2 - \frac{1}{40}t - \frac{3}{8}$$

$$T_{10,3} \sim -\frac{3}{8}t^9 - \frac{1}{40}t^8 + \frac{1}{40}t^7 - \frac{9}{40}t^6 + \frac{9}{40}t^5 + \frac{3}{8}t^4 + \frac{9}{40}t^3 - \frac{9}{40}t^2 + \frac{1}{40}t - \frac{1}{40}$$

$$T_{10,7} \sim -\frac{3}{8}t^9 - \frac{9}{40}t^8 + \frac{9}{40}t^7 - \frac{1}{40}t^6 + \frac{1}{40}t^5 + \frac{3}{8}t^4 + \frac{1}{40}t^3 - \frac{1}{40}t^2 + \frac{9}{40}t - \frac{9}{40}$$

$$T_{10,9} \sim \frac{33}{40}t^9 - \frac{9}{40}t^8 - \frac{3}{8}t^7 + \frac{3}{8}t^6 + \frac{1}{40}t^5 - \frac{17}{40}t^4 + \frac{1}{40}t^3 + \frac{3}{8}t^2 - \frac{3}{8}t - \frac{9}{40}$$

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